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THE UNIVERSITY OF ALBERTA

ENRICHMENT STUDIES IN MATHEMATICS FOR GIFTED STUDENTS IN GRADES NINE AND TEN IN THE SMALL HIGH SCHOOL

by

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A THESIS
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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Enrichment Studies in Mathematics for Gifted Students in Grades Nine and Ten in the Small High School" submitted by Gertrude M. Nelson in partial fulfilment of the requirements for the degree of Master of Education.

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ABSTRACT

Recent emphasis on the need for special provision in schools for the pupil of high ability arises from the belief that there is a loss to the country of the talent of potential future leaders if the child is not challenged to develop his abilities to their fullest extent. Special enrichment studies which are in advance of studies suited to the child of average ability are a means of providing this needed challenge.

In mathematics, enrichment studies should give greater depth and breadth to the student's total experience in the subject. There has been in the past a great lack of suitable enrichment materials in mathematics, a lack which is beginning to ease only in the most recent years.

Enrichment for gifted students in a large high school is usually made possible by placing these students in a separate class so that the teaching can be adjusted to their ability. In the small high school where grouping is not possible the enrichment will have to take the form of individual study and perhaps meetings of a mathematics club or of seminar groups as time permits. A supply of books, materials, and study plans is necessary where there is such an arrangement.

Selecting the gifted students who are worthy of special attention is a matter requiring careful consideration since personality traits and work habits as well as aptitude and mental ability must be taken into account.

In the present thesis the investigator has attempted to prepare a plan for introducing enrichment studies in mathematics for gifted students in a small high school. In line with the criteria as set forth a few units of study are included. These are suitable for individual study by the student with little or no supervision by the teacher. Since the units are particularly designed for use in a school for foreign children in India, they emphasize to some extent the contributions of that country to the field of mathematics.

ACKNOWLEDGEMENTS

The investigator wishes to express sincere thanks to the members of her thesis committee: to Mr. Wm. F. Coulson whose suggestions and comments as supervisor during the entire project were very helpful, and to Dr. L. D. Nelson and Mr. O. Massing for their contributions.

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CHAPTER I

INTRODUCTION

I STATEMENT OF THE PROBLEM

The wide range of abilities among students in the heterogeneous classroom presents a challenge to any teacher. The slower student must be given extra help in order to master the requirements of the particular grade as prescribed in the school system. The bright student with only a little effort does the best work and the overworked teacher has no time to consider what he might do if he were challenged according to his ability. This thesis presents a description of efforts to challenge brighter students in mathematics through special enrichment studies and concludes with some units of study designed for this purpose.

II THE NEED FOR INVESTIGATION

Occasionally in the heterogeneous classroom the proportion of brighter students may be larger than usual. These students reveal, by their questions, comments or activities, an interest that goes beyond the scope of a subject as presented in the classroom. More than likely they are left on their own however as the teacher must give extra time and attention to the few who progress more slowly. Recently it was the

experience of the investigator to work with classes of this nature in a small high school for English speaking foreign children in India. The bright students requested readings in the history of mathematics and, since they had read about them, they were curious about new mathematic programs being introduced experimentally into some classrooms. A few were bored with the classroom diet and did not exert any effort themselves to excel for they could earn satisfactory grades anyway. There seemed to be a need for extra enrichment materials for them to study on their own as they might find time out of class.

III BACKGROUND OF THE INVESTIGATION

Probably teachers have always been aware of individual differences but, in an educational system where certain requirements must be mastered at each grade level, too many teachers accept as their responsibility the task of preparing each student to be able to pass certain examinations designed to measure this mastery of knowledge and thus qualify for promotion. Only a few teachers find it possible to inspire the brighter student to do something beyond what is done in class.

In the early part of the century the usual way of taking care of the gifted was to accelerate them though ability grouping and other forms of special or individual attention did exist also. After 1920 special provision for these students tended to take the form of

enrichment because educators were concerned about the emotional and social development of students if they were removed from their own age group. Often a combination of two or more of these provisions exists in practice. Along with grouping it is necessary to have acceleration or enrichment or both.

The democratic principle of equal educational opportunity for all was responsible for the gradual increase in concern for brighter students as educators came to see that this principle should not be interpreted to mean identical opportunity for all. The child is unique and for each one his educational opportunity should be limited only by his abilities.

> We began to suspect that schools could not serve all children equally well if we simply required all children, ... , to measure up to arbitrarily-established grade standings in an arbitrarily-fixed and uniform curriculum. We began to say--even if we did not always manage to do much about it--that the goal of the modern school is the continuous development of each child, up to the full measure of his potential.[1]

Since about 1950 another reason for growing emphasis on educational opportunity for the gifted is the concern that our countries are losing annually a large proportion of those who have the potential for leadership and for making valuable contributions but who do not develop that potential. Paul Witty is credited with initiating this concern through his book, The Gifted Child. He says "it is estimated

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[1] G. E. Flower, "Promotion Policy," The Canadian School Principal, A. W. Reeves (editor), (Canada: McLelland and Stewart Limited, 1962), p. 303.

that in many states not more than fifty per cent of the total number of gifted students who graduate from high school go on to college."[2] Dael Wolfle's American Resources of Specialized Talent, 1954, revealed shortages of highly trained personnel and resulted in considerable concern about lack of "intellectual resources".[3] Preliminary results of a survey by Holland of 30,000 students who took the National Merit Scholarship Qualifying Test in 1958 show that talent loss is small in the top five per cent of each year's supply of potential college entrants (one out of twenty boys and one out of twelve girls) and greater among those who rank in the next five per cent in ability (boys, one out of ten; girls, one out of five). Altogether about one out of five who rank in the top third of ability and who are interested in full-time college study fail to enroll.[4]

Two factors responsible for recent increased emphasis on special provisions in education for gifted students are the belief that all students should have equal opportunity to develop their abilities to their greatest capacity and the concern about the annual loss of many

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[2] Paul Witty (ed.), The Gifted Child, The American Association for Gifted Children (Boston: D. C. Heath and Company, 1951), p. 268.

[3] Robert F. DeHaan & Robert J. Havighurst, Educating Gifted Children, Revised and Enlarged Edition (Chicago: The University of Chicago Press, 1961), p. 12.

[4] John L. Holland, "The National Merit Research Program," Education, LXXXII (April, 1962), 479.

who have superior ability but who do not fully train or use it because they are not challenged while at school.

IV DELIMITATION OF THE PROBLEM

For this thesis the following indicates the purpose of the investigator and the procedure to be followed in preparing studies for enrichment in mathematics in grades nine and ten in a small residential high school for English speaking children chiefly from the U. S. A. and located in India.

The Purpose

1. Review the literature by exploring existing plans, recommendations and programs primarily as to how they deal with:

    a) Identification of the gifted in mathematics.

    b) Program provisions and topics in mathematics for the gifted.

    c) Experimental findings concerning achievement and also attitudes or reactions toward programs in operation.

2. To prepare topics suitable for individual study by talented students in mathematics in grades nine and ten in a small high school involving:

    a) References and sources.

    b) Study plans for the topics.

The Procedure

On the basis of the literature, the specific school, the average superior ability of the children in the school, and the school's existing regular mathematics courses for grades nine and ten:

1. To set up criteria for the topics mentioned above, criteria dealing with:

    a) Identification of students.

    b) The nature of the studies.

    c) The provisions required of the school.

    d) The requirements for the staff.

2. To set up a series of six topics in line with the criteria, ready for students to follow and containing mostly those items mentioned in the existing plans in the literature.

V DEFINITIONS

The Gifted Child

The gifted student is one who has a high degree of general intellectual ability. DeHaan classifies the top ten per cent of all children as indicated on standard intelligence tests as the gifted of "second-order" while one-tenth of the top one per cent, those at the very top, are the gifted of "first-order" or extremely gifted.[5] Those with an I. Q. above 137 constitute the top one percent of the population,

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[5] DeHaan, op. cit., p. 45.

and for the top ten per cent the cut-off point is an I. Q. rating over 120.[6] Professor Lewis Terman selected children with an I. Q. of 140 or more for his long-range study of gifted individuals. About one-fourth or one-fifth of a high school graduating class are intellectually gifted since they constitute the top ten per cent of the population as a whole.[7]

I. Q. ratings alone cannot accurately identify gifted students worthy of special provision for these are known to fluctuate from year to year.[8] Furthermore intelligence is composed of many factors. Two students may have the same I. Q. One of them may have high verbal and reasoning ability but low number and spatial ability and the other may have the reverse. DeHaan says that differential aptitude tests, social contacts, achievement tests, personality inventories and tests, and teachers' observations and recommendations all have a part in the identification.[9]

The term talented is frequently used in place of the term gifted by those who make no distinction between the two words. Others say that the talented are those with specialized ability but many of these add that the person talented in some specific way is first gifted

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[6] Education Policies Commission, Education of the Gifted, (Washington: National Education Association of the United States, 1950). p. 43.

[7] *Ibid.*, p. 44.

[8] DeHaan, *op. cit.*, p. 41.

[9] *Ibid.*, p. 45.

intellectually. The terms gifted and talented have the same meaning in this thesis; furthermore, the bright, the superior, and the able are also used in place of the gifted.

Enrichment, Acceleration, and Ability Grouping

For the three plans for special provision for the gifted--enrichment, acceleration, and ability grouping--definitions as given by different writers are not all the same. Passow[10] says the three plans are administrative rather than instructional. They stress differences in class organization rather than in what goes on in the classroom. Goldberg[11] lists ability grouping and acceleration as administrative procedures and enrichment as a teaching method and course content problem. She asserts that we really don't know what enrichment is, for everyone talks about it but few do anything about it. Hall[12] regards enrichment as a teaching procedure and acceleration as an administrative arrangement.

Enrichment. Enrichment may be said to consist of "practices"

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[10] A. Harry Passow and others, Planning for Talented Youth, (Bureau of Publications, Teachers College, Columbia University, 1955), p. 36.

[11] Miriam L. Goldberg, "Research on the Talented," Teachers College Record, LX (December, 1958), p. 156.

[12] Lynn G. Hall, "A Review of Acceleration Procedures for Gifted Pupils in Regular and Special Classes," Alberta Journal of Educational Research, IV (March, 1958), p. 50.

______, "A Review of Enrichment Procedures for Gifted Pupils," Alberta Journal of Educational Research, IV (June, 1958), p. 117.

such as special assignment, independent study, individual projects, and small group work "which are intended to increase the depth or breadth of the gifted students' learning experiences. This purported form of provision for the gifted often in fact merely camouflages do-nothingness."[13] Hall defines enrichment as a teaching procedure using a variety of resources and experiences to give a new and richer meaningfulness to life. It entails the selection and organization of learning experiences aimed at the optimal development of the child.

Vertical enrichment or enrichment in depth is accomplished by providing for specialization or more mature work in a given area. The child can pursue studies in a field or practice a certain skill until he is satisfied. Acceleration, along with enrichment, is possible here. Horizontal enrichment or enrichment in breadth gives a student opportunity to work in areas not normally included in the child's school experience. He can take in broader interests and "assimilate and integrate complex ideas."[14] Here there is no acceleration. The former is in advance of the grade level but the latter is at the same grade level.

Present in most definitions of enrichment are these two thoughts, --optimal development of the abilities of the individual

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[13] The Gifted Student: A Manual for Program Improvement, A Report, Southern Regional Project for Education of the Gifted (Atlanta, Georgia: Southern Regional Education Board, 1962), p. 30.

[14] DeHaan, op. cit., p. 81-82.

and depth and breadth in learning experiences. These essentially explain enrichment as it is used in this thesis. Enrichment does not mean the assignment of more difficult work of the same type but it does mean activity that adds to or goes beyond the regular studies.

Acceleration. "An administrative arrangement that implies grade skipping or progress through sequential but rapidly paced learning."[15] "It is any administrative practice designed to move the student through school more rapidly than usual. It includes such practices as early admission, grade skipping, advanced placement, telescoping of grade levels, credit by examination, etc."[16]

Ability Grouping. One definition of ability grouping states that it is "the practice of assembling or deploying students for instructional purposes who are somewhat nearer together in general capacity for learning or in given specific aptitudes, so that instruction and learning may proceed at a pace and in terms of qualities suited to this (these) capacities."[17]

The Small High School

The high school having an enrollment of 200 or fewer in grades nine through twelve shall be defined as a small high school.

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[15] Hall, *loc. cit.*

[16] *The Gifted Student: A Manual for Program Improvement*, *op. cit.*, p. 29.

[17] *Ibid.*

VI SUMMARY

The problem of the need for making some provision for challenging the gifted student according to his ability, in the heterogeneous classroom of the small high school, has been presented in this chapter. The plan for preparing enrichment units for bright students in mathematics in grades nine and ten has been outlined and some important terms have been defined. The studies will meet certain criteria and there will also be criteria established for the school, the students, and the staff. The units will be used for individual study by gifted students in a school for foreign children in India.

CHAPTER II

THE REVIEW OF THE LITERATURE

Three aspects of the education of gifted children contained in the literature are reviewed in this chapter: the identification of the gifted, descriptions of enrichment projects in mathematics, and experimental studies in enrichment in mathematics. A great deal has been written about the first and there exist several descriptions of enrichment projects, but there is very little in the literature on experimental studies of the type mentioned here.

Descriptions of enrichment projects include topics suitable for individual study by the student rather than enrichment curricula to be used for classroom teaching to a superior group. Experimental studies reviewed deal with acceleration and grouping as well as with enrichment as there are scarcely any experiments conducted in enrichment alone.

I IDENTIFICATION OF THE GIFTED IN MATHEMATICS

Identification Through Observation of Characteristics

The literature stresses the importance of accurate selection of those with superior ability almost more than it stresses special provisions for the gifted. There are students who are gifted generally and there are those who are gifted or talented in special fields such as music. Therefore, the information needed is varied in nature.

There is also emphasis on the need for identifying giftedness early in childhood, that is, during the years in elementary school, recognizing at the same time that tests are not infallible and that there are sometimes "late-bloomers."

Scheifele's compiled list of intellectual characteristics of the gifted is probably the most complete.

> Intellectually, the gifted child, in relation to other children, tends to:
>
> A. Possess superior ability in reasoning, generalizing, dealing with abstractions, comprehending meanings, thinking logically, and recognizing relationships.
> B. Perform highly difficult mental tasks, an ability described as "power."
> C. Learn more rapidly and easily.
> D. Show intellectual curiosity.
> E. Possess superior insight into problems.
> F. Have a wider range of interests.
> G. Show greatest superiority in reading ability, both in speed and comprehension; language usage; arithmetical reasoning; science; literature; and the arts.
> H. Do effective work independently.
> I. Apply originality and initiative in intellectual tasks.
> J. Show less patience with routine procedures and drill.
> K. Exhibit alertness, keen observational ability, and quick response.
> L. Show as much unevenness in abilities in the subject-matter areas as other children.
> M. Have a longer interest span; show more interest in abstract than practical subjects; exhibit greater superiority in attainment in abstract subjects and less in manual activities.
> N. Have an interest in the future, a concern with origin, destiny, and death though unable emotionally to accept realities of the latter.[1]

In the report of the Secondary School Curriculum Committee of the

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[1] Marian Scheifele, The Gifted Child in the Regular Classroom, (New York: Bureau of Publications, Teachers College, 1953), p. 6.

National Council of Teachers of Mathematics, general characteristics and also special characteristics of the gifted in mathematics are indicated. The latter are:

(a) Recognizes patterns readily and enjoys speculating on generalizations.
(b) Prefers to think on higher levels of abstraction.
(c) Classifies particular cases as special cases of more general situations with relative ease.
(d) Follows a long chain of reasoning, frequently anticipating and contributing.
(e) Frequently asks profound questions.
(f) May be reading mathematics books years ahead of his class.
(g) Is frequently impatient with drill and details that he thinks are not important.[2]

Fehr mentions the lack of adequate means of identifying giftedness other than through formal tests. He describes the gifted child as "one who shows to an exceptionally high degree the ability to work with ideas. He is exceptionally capable in thinking, that is in the manipulations and creation of abstractions of work and number."[3] While these may describe academic giftedness he points out that there are also other special types of giftedness,--mechanical, social and musical.

Fehr's list of characteristics includes:

extraordinary memory; ability to do abstract thinking at a high

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[2] The Secondary Mathematics Curriculum, Report of the Secondary School Committee (Washington, D.C: National Council of Teachers of Mathematics, May, 1959), p. 410.

[3] Howard F. Fehr, "General Ways to Identify Students with Scientific and Mathematical Potential," The Mathematics Teacher, XLVI (April, 1953), p. 230.

> level (make generalizations quickly); the ability to apply knowledge; intellectual curiosity; persistent goal-directed behavior; intuition, insight and penetrability in a problem; a hobby which is all-absorbing; virtuosity; and sound knowledge of advanced areas.[4]
>
> The mark of giftedness lies not in possession of these characteristics, but in possessing them at a high deviation from even the more able students.[5]

Johnson says descriptions of the gifted always emphasize intellectual curiosity, ingenuity, independence, high reading interest, imagination, creative talent, leadership, and ability to assimilate and generalize.[6] In another recent article he speaks of the creative student in mathematics as a nonconformist (sometimes unruly) who is

> unpredictable, flexible, versatile in responses to situations, ideas, problems
> adaptive in association, redefinitions, reorganization and elaborations of ideas
> curious about ideas, objects, devices
> sensitive to problems, relationships, errors
> independent and confident in his judgement and approach
> able to sustain uncertainty and withhold decisions in a complex situation
> able to abstract generalizations from complex situations or apply generalization to a new situation.[7]

He adds that as far as he knows there is no published test which will identify a student having these characteristics and observation is

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[4] Ibid., pp. 231-234.

[5] Ibid., p. 231.

[6] Donovan Johnson, "Lets Do Something for the Gifted in Mathematics," The Mathematics Teacher, XLVI (May, 1953), p. 322.

[7] Donovan Johnson, "Enriching mathematics instruction with creative activities," The Mathematics Teacher, LV (April, 1962), p. 239.

the method that must be used.

While creativity and giftedness are not identically the same, Johnson's article is of interest at this point and his topic is one that is receiving more attention at the present time. DeHaan thinks of creativity as the quality that tends to the production of something new and desirable. He says it can be taught and that every child should experience the thrill of creating new things but in particular the gifted child because he will eventually contribute to all of society.[8] Johnson thinks that to be creative requires more than high intelligence, special talent, or skill, though it is related to possessing intellectual skills.

In identifying the gifted through observation, teachers can be much in error. Fehr states that in certain studies only fifteen out of one hundred selections were correct when the gifted were identified in this manner.[9] A study by Lewis, described by DeHaan, showed that in a large sample of students tested with various standardized tests, teachers' judgements as to those who were at the genius level totalled 341, using their own interpretation of the criterion genius. Of these 341, forty per cent had I. Q.s of 120 and above, the remaining sixty

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[8] Robert F. DeHaan & Robert J. Havighurst, Educating Gifted Children, Revised and Enlarged Edition (Chicago: The University of Chicago Press, 1961), pp. 166-67.

[9] Fehr, *op. cit.*, p. 231.

per cent had I. Q.s from 119 to below 70.[10] However, until there are more objective methods of recognizing and detecting characteristics of giftedness in children, observation which involves the cooperation of principals and teachers will continue in practice.

Identification Through Testing

The inaccuracy of identifying the gifted by observation has been pointed out as has also the lack of suitable tests for detecting characteristics of giftedness. The detection of superior mental ability through intelligence and aptitude tests and inventories has already been mentioned in the previous chapter and will be referred to again in Chapter III.

II DESCRIPTIONS OF PLANS AND PROVISIONS

Enrichment and its Variations

The methods of making possible studies of greater "depth and breadth" through enrichment may be in the form of individual research following an outline, projects, instruction in small groups in the classroom, or individual assignments. There may be small group seminars in or out of class, correspondence lessons, mathematical clubs, contests and sometimes fairs. Mostly the activities will be carried on outside of class time, but in some arrangements brighter students finish class

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[10] DeHaan, op. cit., p. 53.

work quickly and are released for some minutes of class time to pursue extra studies.

Jackson reports the response to a questionnaire sent in April, 1958 to mathematics departments of the Minneapolis junior and senior schools.[11] Provisions made for enriching the program of the superior pupil in regular classrooms are in the nature of special assignments:

doing library research work
reading for interest or recreation
writing book reports
giving oral reports
solving additional and more difficult problems
investigating topics of special interest
making models
preparing exhibits for science fairs
making scrapbooks
watching and reporting on mathematics television programs
organizing and conducting meetings of a mathematics club

Among additional activities reported for the superior pupil in the senior high school were:

preparing lessons to teach to their classes
conducting class discussions
writing mathematics term papers
meeting in groups to review for college entrance examinations
competing in the national mathematics contest
attending special Saturday morning classes at the University of Minnesota

Donovan Johnson comments favorably on differentiation in assignments as "probably the simplest and often the most practical method of providing for the superior student."[12] He speaks of an

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[11] Harvey O. Jackson, "The superior pupil in mathematics," The Mathematics Teacher, LII (March, 1959), p. 203.

[12] Donovan Johnson, "Lets Do Something for the Gifted in

enriched curriculum as possible during the school day, during night school or through correspondence courses. However, individualized instruction or group instruction within the classroom is rarely practiced by the over-burdened teacher though the literature recommends these methods.

Both Scheifele[13] and Witty[14] mention the dearth of enrichment programs and the fact that they seldom appear in the literature. Jackson says that "the preparation of such materials is a project that needs to be worked on."[15] Smaller classes, more free time, and the services of a teacher-consultant are often the requirements named as necessary for the teacher who seeks to initiate an enrichment program for the gifted in the regular classroom. A fourth requirement, the availability of suitable materials, is one which, if met, should ease the strain on the over-worked teacher trying to find and prepare materials himself. Wirszup also admits that the stock of enrichment materials in the U.S.A. is low and describes the rather rigid yet fairly successful system of promoting enrichment in mathematics for all students in the schools of East Europe. Mathematicians (professors and

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Mathematics," The Mathematics Teacher, XLVI (May, 1953), p. 322.

[13] Scheifele, op. cit., p. 41.

[14] Paul Witty (editor), The Gifted Child, The American Association for Gifted Children (Boston: D. C. Heath and Company, 1951), p. 273.

[15] Jackson, op. cit., p. 204.

research workers) work along with teachers in preparing a varied supply of both enrichment materials and books. The materials contain problems and recreational mathematics within the scope of the high school curriculum and the books include aspects of higher modern mathematics and new mathematical theories. He recognizes the difficulty of writing such books and then makes this statement.

> Note, finally, how neatly our country's needs and interests in school mathematics can be focussed in the area of enrichment. Contributions from specialists can take the form of enrichment materials designed for both teacher and student. The teacher's concern for the revision of the curriculum and improvement of mathematical pedagogy can begin to find expression in the form of experiments with enrichment materials. In this way enrichment can serve as the cutting edge of improvement in mathematics and of its articulation with higher mathematics.[16]

Content and Nature of Enrichment Programs in Mathematics for Individual Study

Descriptions of efforts to stimulate the gifted student to explore and extend his grasp of mathematics are usually in the form of outlines or plans designed by the teacher or a group in the local school system and aided by a counsellor.

The following appears in a small bulletin of the National Council of Teachers of Mathematics.[17] The topics are suitable for the first two

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[16] Izaak Wirszup, "Some remarks on enrichment," The Mathematics Teacher, XLIX (November, 1956), p. 525.

[17] E. P. Vance (editor), Program Provisions for the Mathematically Gifted Student in the Secondary School (Washington, D.C: National Council of Teachers of Mathematics, 1957), p. 10.

years of high school but a single class would not study all the topics at that level in one year.

1. In Elementary Algebra (Ninth Year):
    a. Second order determinants used to solve simultaneous linear equations in two unknowns;
    b. Logarithms used in the exponential form;
    c. Use of the transit and sextant in connection with the unit on numerical trigonometry;
    d. A simple unit on descriptive statistics using either class or school statistics gathered and analyzed by the class;
    e. A study of the linear function by means of slope and intercepts and making use of the delta notation to serve as a foundation for the introduction of calculus later in the high school;
    f. A simple unit from number theory using the concepts and notation for congruences;
    g. A simple unit in demonstrative geometry;
    h. Historical sidelights illustrating the development of some mathematical idea, or the biographies of great mathematicians.

2. Demonstrative Geometry (Tenth Year);
    a. The three famous problems of antiquity;
    b. More difficult constructions;
    c. A unit on coordinate geometry;
    d. Logical reasoning in "life situations";
    e. Non-Euclidean geometries;
    f. Constructibility of regular polygons;
    g. Integral Pythagorean triangles;
    h. Extensions of geometric concepts into three (and more) dimensions;
    i. Conic sections;
    j. Historical sidelights.

Payne[18] developed twenty-four topics suitable for enrichment of the regular first and second year high school algebra courses and which the bright students might be able to use on their own with very little

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[18] Joseph N. Payne, "Self-instructive enrichment topics for bright pupils in high school algebra," The Mathematics Teacher, LI (February, 1958), pp. 115-16.

help from the teacher. These included:

1. Algebra in Retrospect (history)
2. Approximate Numbers (precision, accuracy and error, relative error)
3. Congruence of Numbers
4. Cube Roots of a Number
5. Domain and Range of a Function
6. Instruments for Indirect Measure (clinometer, hypsometer, carpenter's square)
7. Magic Squares
8. Mathematical Recreations and Puzzle Problems
9. Men of Mathematics
10. Number Systems Other than Ten
11. Postulational Algebra
12. Prime Numbers (definition, Sieve of Eratosthenes, Fermat's formula)
13. Pythagorean Relationship (integral solutions)
14. Rate of Change (constant change, rate of change of a function, of statistical data)
15. Scientific Notation
16. Short Cuts in Computation
17. Slide Rule
18. Arithmetic Mean
19. Mean and Standard Deviation
20. Normal Curve of Probability
21. Permutations
22. Combinations
23. Binomial Expansion (Pascal's Triangle)
24. Probability.

For students of grade ten in Cincinnati who were already preparing to study the Advanced Placement Mathematics of the College Entrance Examination Board, Kieffer[19] lists individual projects and assignments which were completed. The subjects were: the nine-point circle, harmonic division, special constructions, golden section, angle

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[19] Mildred Kieffer, "Meeting the Needs of Cincinnati's Gifted Pupils in Mathematics," New Developments in Secondary School Mathematics, Bulletin of the National Association of Secondary School Principals, 44 (May, 1959), p. 90.

division devices, fallacies and non-Euclidean geometries.

Dickie[20] lists some of the fifty-two topics he prepared as supplementary topics for grades seven and eight for brighter students, and others if they desired, to develop after their work was completed in class or outside of class time. Some of these were:

Secret of the Magic Number Chart (binary number system)
Egyptian Number Symbols and Fractions
Duodecimal Number System
Multiplication Using Lattices
Using New Symbols
Russian Peasant Multiplication
Using Induction in Series
Pascal's Triangle
Figurate Numbers
Induction in Geometry
Complex Fractions
Achilles and the Tortoise
Repeating Decimals
Some Peculiar Things About Zero
Order of Operations
Old Russian Numerals
Binary Addition and Subtraction
Operations Using Roman Numerals
Tidas and His Crows (inductive reasoning)

Johnson[21] describes the creative student or original thinker in the classroom as a nonconformist who does not necessarily rate unusually high on an intelligence test. He suggests that "the work of a mathematician as he searches for new knowledge should give examples of creative activities" and that by investigating their work a student

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[20] Paul Dickie, "A supplementary program in junior-high-school mathematics," The Mathematics Teacher, LV (January, 1962), p. 58.

[21] Donovan Johnson, "Enriching mathematics instruction with creative activities," The Mathematics Teacher, LV (April, 1962), pp. 238-42.

could be inspired to investigate original ideas. For each of the following an important incident or contribution is suggested: Galois (group theory); Archimedes (determination of the value of π); Eratosthenes (circumference of the earth); Euler (topology); Gauss (sum of an arithmetic series); Fermat's last unsolved theorem; Newton (calculus); Goldback (primes); Descartes; Pythagoras (musical scale); Leibniz (binary numbers); Einstein; Cantor (sets); Riemann and Lobachewski; and Von Neumann (game theory). Original work which a student might then do would be to invent new symbols and a new numeration system; invent new operations or new ways to perform divisions, multiplications, or the finding of roots; discover a new proof for a theorem; invent a new scheme of measurement with appropriate units and measuring devices; perform experiments to discover the pattern in the results obtained; develop a cylindrical geometry, a geometry for the sphere, a geometry for a finite plane or convex region; solve an original problem by linear programming; or devise a finite mathematical system.

Several articles describe special summer institutes held usually on a college campus for a period of six weeks. Generally the students are juniors, with sophomores or freshmen sometimes included, who have been selected on the basis of three or four criteria such as I. Q. rating, achievement on standardized or other selection tests, contest results, transcript, an essay or letter written by the student, and recommendations of principal and staff. Lectures are given by college

professors who seek to give insight into the nature and scope of modern mathematics, emphasizing concepts rather than skills. In addition the students carry out research on associated topics, culminating in a written report, an oral report, or a class discussion.

At Rice Institute students' reports included: (1) the Jordan curve theorem for polygons, (2) the coloring of maps, (3) Euclid's and special cases of Direchlet's Theorems of the infinitude of primes in arithmetic progressions, (4) network problems and the bridges of Koenigsberg, (5) Euler's formula and the regular polyhedra, (6) perfect numbers and Mersenne primes, (7) extremal problems and the isoperimetric inequality, (8) paradoxes and the logical antimonies. Some longer student reports were on the topics: linear algebra and systems of linear equations; polynomial equations of the second, third, and fourth degrees; probability; games of strategy; and groups, rings, and fields.[22] In addition Nichols lists diophantine equations, perfect numbers and congruences, programming techniques for an electronic computer, quantifiers, and statement calculus (truth tables) in his report of a summer institute at Florida State.[23]

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[22] L. K. Durst, "The Rice Institute Summer Program for Talented High School Students," New Developments in Secondary-School Mathematics, Bulletin of the National Association of Secondary-School Principals, 44 (May, 1959), p. 75.

[23] Eugene P. Nichols, "A Summer Mathematics Camp for Talented High School Students," New Developments in Secondary-School Mathematics, Bulletin of the National Association of Secondary-School Principals, 44 (May, 1959), p. 102.

Several summer institutes sponsored by the National Science Foundation are reported in The Mathematics Teacher in 1962. The list of students' topics is compiled from four articles.[24]

prime numbers
numeration systems
symbolic logic
multivalued logic
George Boole's mathematical analysis of logic
miniature geometries
congruence and modular systems
repeating decimals
game theory
vectors and matrices
the abacus, the sextant
lattices
linear programming
Greek mathematics
the trisection problem
non-Euclidean geometry; Lobatchewskian geometry
attempts to prove the parallel postulate
Pythagorean triples
topology
development of mathematics (history)
probability; probability as applied to Genetics
the design of a simple digital computer
computer addition of binary numbers

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[24] Leslie A. Dwight, "A summer program in mathematics for high ability secondary students," The Mathematics Teacher, LV (March, 1962), pp. 179-83.

Clifford Bell, "A summer mathematics training program for high-ability secondary-school students," The Mathematics Teacher, LV (April, 1962), pp. 276-78.

Wayne W. Gutzman, "A description of a summer institute for academically talented high-school juniors," The Mathematics Teacher, LV (April, 1962) pp. 279-81.

Rev. Alfons J. Van der Linden, "A summer mathematics program for high-ability secondary-school students," The Mathematics Teacher, LV (May, 1962), pp. 369-77.

group theory
principles of cardinality and the transfinite cardinal numbers
mathematical induction
inequalities
inversion
infinity
Isaac Newton
conic sections
spherical trigonometry
ordered triples (a Westinghouse project)

It should be noted that topics as used in classrooms mentioned previously and in the summer institutes are not all suitable for the grade nine and ten levels. In fact one concludes from the foregoing descriptions that what is suitable for enrichment at a certain grade level is, in part, a matter of opinion. Furthermore, the students at the summer institutes would not compare with students in a regular classroom since they were a select homogeneous group attending lectures on related topics given by specialists, having access to special materials, and studying under supervision.

Large Scale Mathematics Programs

In recent years various committees have attempted to rewrite mathematics curricula for the school grades in an effort to "up-date" them. The basic content of traditional courses is, they say, the same as that taught one hundred years ago or more. Discoveries since that time have brought great advances in mathematical knowledge and members of these committees feel that certain concepts of modern mathematics can be grasped by children. By bringing these into school programs there is greater unity in the mathematics curriculum, and, in keeping

with the true nature of mathematics, understanding of concepts becomes more important than mastery of skills. Some of the unifying ideas are: structure; operations and their inverses; measurement; extensive use of graphical representation; systems of numeration; properties of numbers, development of the real number system; statistical inference, probability; sets--language and elementary theory; logical deductions; valid generalizations.[25]

There are some differences among the various committees depending upon the grade level for which their materials are prepared and whether these materials are suitable for superior children only or for all pupils. They differ also in membership and sponsorship. Three of the new programs are briefly described here.[26]

The University of Illinois Committee on School Mathematics. Beginning in 1952 the University of Illinois Committee on School Mathematics (UICSM), under the leadership of Dr. Max Beberman, has prepared materials for college-capable students of grade nine, ten, eleven, and twelve. Understanding of principles through discovery and precision of terminology are emphasized. There have been revisions after first experimenting with the materials in classrooms.

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[25] The Revolution in School Mathematics, A Report (Washington, D. C: National Council of Teachers of Mathematics, 1961), p. 22.

[26] Bruce E. Meserve & Max A. Sobel, Mathematics for Secondary School Teachers (Englewood Cliffs, N. J., Prentice-Hall, Inc., 1962), pp. 2-6.

Julius H. Hlavaty (editor), Mathematics for the Academically

The University of Maryland Mathematics Project. The University of Maryland Mathematics Project (UMMaP) has produced materials for grades seven and eight again emphasizing precise vocabulary, structure, numeration systems, and introduction to geometry. Dr. John R. Mayer is the leader of the project which began in 1957.

The School Mathematics Study Group. Mathematicians, teachers, and psychologists are members of the largest nation wide program for mathematics curriculum improvement, the School Mathematics Study Group (SMSG), which began work in 1958. Materials have been prepared, tested in classrooms, and revised for grades seven through twelve, and the committee has more recently set up materials for grades four through six. Geometry has an intuitive approach and some solid geometry and even analytic geometry have been introduced into the course in plane geometry. Algebra and geometry are more integrated. Vectors and matrices appear in the senior courses. Materials were prepared first for the more able students (upper fifty per cent) but there are also SMSG materials for the average non-college-bound student. Students'

Talented Student, A Report, National Education Association Project on the Academically Talented Student and National Council of Teachers of Mathematics (Washington, D. C: National Council of Teachers of Mathematics, 1959), pp. 30-31.

Studies in Mathematics Education, A brief Survey of Improvement Programs for School Mathematics (Scott Foresman and Company, 1960), pp. 10-12, 32-35, and 59-65.

The Revolution in School Mathematics, op. cit., pp. 17-19.

books, teachers' manuals, and monographs for enrichment purposes are available. Dr. E. G. Begle is the leader of the group which began its work at Yale University and now continues at Stanford University.

Many schools throughout the continent have adopted or are preparing to adopt a curriculum that includes modern mathematics. In this thesis it is not intended that these be thoroughly reviewed. They are mentioned in order to point out that what Wirszup was predicting is actually taking place.[27] Many of the topics, considered suitable for enrichment of the traditional course ten years ago are now becoming part of the regular course for all students. In Program Provisions for the Mathematically Gifted Student there is the comment,--"it will be obvious to anyone reading these lists of optional and enrichment topics that many of them have attained standing in standard courses of study."[28] DeHaan notes with more reserve that the new programs are being tried on the gifted and that there is some question as to whether they will change mathematics teaching for average children to any extent.[29] Keeping in mind the meaning of enrichment one cannot suggest that it will cease to be important for the gifted student, but that, in a few years and in a variety of ways depending upon the school system, it will contain added content replacing that which is becoming a part of regular

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[27] see p. 20.

[28] E. P. Vance, op. cit., p. 10.

[29] DeHaan, op. cit., p. 87.

courses.

The Advanced Placement Program

One more nation wide plan, an acceleration plan designed for college-bound students, must be mentioned. The Advanced Placement Program of the College Entrance Examination Board is a plan whereby the gifted student is permitted to progress into college level mathematics before he graduates from high school. He accelerates through the standard high school mathematics studies in three years and possibly a summer session in addition so that he may study college freshman analytic geometry and calculus in his senior year at school. The manner and time of beginning the accelerated program will vary from school to school. In the senior year the student takes the Advanced Placement Examinations, results of which are sent to the college he will attend. The college reserves the right to award him advanced standing or credit or both and, if granted, he proceeds to advanced studies in mathematics in his college freshman year.[30]

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[30] Edwin C. Douglas, "The Advanced Placement Program of the CEEB," New Developments in Secondary-School Mathematics, Bulletin of the National Association of Secondary-School Principals, 44 (May, 1959), pp. 92-95.

The Revolution in School Mathematics, op. cit., p. 49.

Studies in Mathematics Education, op. cit., pp. 41-42.

III RESEARCH STUDIES IN ENRICHMENT AND OTHER PROVISIONS FOR THE GIFTED

What benefit is there for a gifted student who has had opportunity in enrichment studies in mathematics as compared with the gifted student who has had no such opportunity? The few studies that do make some attempt to answer this question use achievement results as the means of determining any benefit but there are some that take into account improved attitude or greater appreciation for mathematics as a desirable outcome of enrichment.

Eight of the studies that follow are of the first type, only three of these dealing with enrichment alone. After these are four studies which attempt to measure the effect of enrichment on attitude.

Experimental Studies of Achievement in Mathematics

Albers and Seagoe[31] prepared units of enrichment study for the second semester of grade nine algebra designed to broaden and deepen the knowledge of the content of the regular course. Students with an I. Q. of 120 or more, who were members of the ten first year algebra classes of four teachers willing to participate in the study, made up two groups equated on the basis of I. Q., initial algebra achievement (Algebra Cooperative, Form S), chronological age, and mean intelligence. Though the mean intelligence favoured the control group slightly, the writers

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[31] Mary E. Albers & May V. Seagoe, "Enrichment for Superior Students in Algebra Classes," Journal of Educational Research, (March, 1947), pp. 481-95.

say this was offset by the "slightly greater time the others had for independent study." There were seventeen boys and fifteen girls in each group. The experimentals, who were informed that they had been selected for special study because they were capable, used fifteen per cent of their normal algebra class time plus certain additional alloted time for their study or discussion of enrichment topics following the outlines provided.

Results of the Algebra Cooperative Achievement (Form T) test given at the end of the semester showed no significant difference but the improvement on a test on knowledge of enrichment materials was significantly greater for the experimental group. For the grade nine classes involved it was concluded that the superior student class time on regular work can be reduced without loss of achievement. If time were to be considered, one might wonder about the results if the control group had spent as much time on traditional study as the experimentals spent in total on regular study plus enrichment.

Long[32] studied ninety-eight eleventh graders to see if a curriculum enrichment program in advanced algebra brought significant gains for talented students in the heterogeneous classroom. He also sought to determine whether enrichment in the regular classroom affected the educational achievement of the non-talented and to see if the

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[32] Roy Gilbert Long, A Comparative Study of the Effects of an Enriched Program for the Talented in Advanced Algebra Classes, Doctoral Dissertation (Indiana University, 1957).

program brought any changes in attitude for either the experimental or the control group. Fifteen of the fifty-four experimentals in Group A were talented and eleven of the forty-four in the control group, Group B, were talented. The talented were those with a minimum combined pretest score of 190 on the Davis Test of Functional Competence in Mathematics Achievement and the Cooperative Intermediate Algebra Test. All studied second course algebra as outlined in the course of study for Indiana, and in addition Group A participated in enrichment activities such as supplemental reading and reports, special projects, serving as group leaders, working review lists for extra credit, participating in mathematics contests, and giving discussions on new materials and topics.

Differences in pretest and posttest scores on the Davis and Cooperative tests were analysed. These were significant in favour of Group A on the Cooperative test and on the combined test scores for Groups A and B and also for the talented of these two groups. Net gains on the Cooperative Test were significant in the same instances and were almost significant for the non-talented of Groups A and B. Student reactions of the experimental group were slightly more favourable toward the study of mathematics than were those of the control group.

Long concluded that enrichment materials were proved to be a worthwhile educational technique though the experiment cannot be claimed to be representative of the general population.

The top sixteen students chosen from the top ninth grade classes

in a California junior high school were taught algebra three times a week and during the remaining time usually given to mathematics prepared reports, held discussions, went on field trips, or participated in certain college lectures. Three years later Wilson[33] examined them to "see whether shortened periods of time in ninth grade algebra was harmful to further work in mathematics" and to "see whether seminar periods in place of algebra periods produced measureable achievements in academic work other than mathematics." Using previous records for eighth grade he chose a "comparison" group at the end of the three years. There was a significant difference in favour of the experimental group in the first two of the four cases: I. Q. ratings, California Achievement for Arithmetic, California Achievement for Reading, and Lee Algebraic Test.

When data from the end of the three years were compared with previous records the following items showed no significant difference: seventh and eighth grade arithmetic and tenth grade geometry, eighth grade arithmetic and tenth grade arithmetic, eighth grade reading and tenth grade reading, and academic grades except mathematics for seventh and eigth grades and those for tenth, eleventh and twelfth grades

It was stated that "though the experimental group had had less time for the regular algebra in ninth grade there is no difference in

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[33] J. A. R. Wilson, "Some Results of an Enrichment Program for Gifted Ninth Graders," Journal of Educational Research, LIII (December, 1959), pp. 157-60.

their achievement as compared with the comparison group," but it should be noted that the two groups were not matched originally. It is also possible that the groups had not had the same mathematics courses in the years following grade nine.

Passow and Goldberg[34] review the Cheltenham Mathematics Study providing enriched mathematics for gifted junior high school students through ability grouping. Four groups of seventh graders, matched according to I. Q. (132-133), arithmetic achievement (within six months), age (within six months), and sex, were taught by teachers who compared to within three categories on a rating scale. The teachers, who participated by invitation, had classes of their choice and took part in planning sessions. Following the normal plan for Cheltenham, the control group studied arithmetic during grades seven and eight, began elementary algebra late in grade eight, and before the end of grade nine started the second year of algebra. The acceleration group studied seventh and eighth grade arithmetic and started ninth grade algebra while in grade seven, finished this and started second year algebra in grade eight which they completed in their ninth year. There were two enrichment groups, one of whom studied units of the University of Illinois Committee on School Mathematics (the teacher had had orientation during the summer) and the other continued with other

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[34] A. Harry Passow, Miriam L. Goldberg, & Frances R. Link, "Enriched Mathematics for Gifted Junior High School Students," Educational Leadership, XVIII (April, 1961), pp. 442-48.

enrichment units. These groups carried on in these plans during grade nine except that the second group used College Entrance Examination Board material for enrichment in place of locally planned units.

Significant differences in results of the STEP tests can be indicated in tabular form where the method or approach named in the left margin is favoured in each case as shown by the asterisk:

	First Enrichment Group (UICSM)	Second Enrichment Group	Control
Accelerated Group		*	*
First Enrichment Group (UICSM)			*
Second Enrichment Group			

The same results show up for a teacher-prepared test of twenty-four items, six for each approach, given in order to test on the modern mathematics learnings not included in the STEP test.

The experiment indicates that at least for gifted junior high students in the schools involved there will be greater mathematical competence either through increasing the pace of the usual mathematics course or through introducing additional more difficult content. If enrichment topics are not more advanced or more difficult the results do not obtain.

Hartung[35] reports an experiment involving twenty-five grade nine

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[35] Maurice L. Hartung, "High School Algebra for Bright Students,"

students of the Laboratory School of the University of Chicago. The mean I. Q. of the group was 141.4, range, 120 to 184. The class as a group had been selected during grade seven in connection with a study by the department of English and, except for a few changes, had remained as a group.

There had been a slight introduction to statistics during seventh grade and in grade eight they were permitted to move as fast as they were able. In the second semester they began first year algebra which they completed in the first semester of grade nine. It was then decided to proceed, using a more experimental approach, to some parts of intermediate or eleventh grade algebra. Along with other topics they studied functions, linear, conic and exponential, often using original data so that they had to find the formula of a given type which would give the "best fit." They considered deviations and the "method of averages," and studied the effects of changes of a and b in a general equation, say $y = ax + b$. Standardized test results (see table below) indicated that

		Mean	Percentile
Cooperative Elementary Algebra Test	Beginning of grade nine	64.2	88
	End of grade nine	70	96
Cooperative Intermediate Algebra Test	End of grade nine		62 for median

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The Mathematics Teacher, XLVI (May, 1953), pp. 316-21; 325.

the pupils had apparently grasped many concepts of the course though several of the ordinary topics of intermediate algebra were not included. The gain on the first test as a result of the year's work was significant at less than the one per cent level.

Hartung concludes:

> bright students can, at an early age, learn much more mathematics than they are usually given an opportunity to study. The hypothesis that as a rule they are not being fully challenged has received some additional experimental verification. There is no evidence that the material these students were taught was the best for them at their level of advancement, nor that other students of lower ability could succeed with the same sort of work.[36]

Justman[37] reports concerning more than seventy matched pairs of junior high students where the experimentals in a special progress class were accelerated through grades seven, eight, and nine in two years. On Cooperative tests in mathematics, science, and social studies there were significant differences in favour of the special progress group, even after adjusting to remove those items from the tests pertaining to grade nine that the control group had not studied and also after further adjustment in order to equate the groups on initial reading skills.

He concludes that intellectually gifted children can be accelerated by one year during junior high school without experiencing loss in those areas being investigated, and he claims further that the

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[36] Ibid., p. 325.

[37] Joseph Justman, "Academic Achievement of Intellectually Gifted Accelerants and Non-Accelerants in Junior High School," School Review, LXII (March, 1954), pp. 142-50.

advantage that the special progress group had in covering more material (that is, some grade nine) and in initial reading ability both jointly and independently do not wholly account for the superiority in their achievement. His assumption is that no other extraneous factors have influenced achievement.

Some time later Justman[38] consulted records and identified a fairly large group of accelerants in four high schools who had been in special progress classes in junior high school and these he matched with a group of non-accelerants on several items except age as the latter would be one year older. He compared achievement marks in grades ten and eleven where the students had studied the same subjects and found differences in only three out of twenty-five instances, one of these favouring the non-accelerants. Marks on statewide Regents Examinations showed no significant differences.

It appears there is no loss of achievement in grades ten and eleven for students who were accelerated one year during grades seven, eight, and nine as compared with intellectually equivalent students who were not thus previously accelerated and who are one year older on the average.

Under supervision of the Minnesota National Laboratory various experiments have been conducted beginning in 1959 using SMSG materials.

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[38] Joseph Justman, "Academic Achievement of Gifted Accelerants and Non-Accelerants in Senior High School," School Review, LXII (November, 1954), pp. 469-73.

Using methods of stratification, the experimenters obtained teachers of varied qualifications and schools from cities, towns and rural areas. In one school,[39] where there was an experiment in grouping, the top twenty per cent or seventy pupils of seventh grade were randomly divided into experimental and control groups, the experimentals remaining as a group and the controls dispersed among other grade seven classes. For the first year traditional courses were used and in the year following they used the SMSG course.

No significant difference in achievement appeared between experimental and control groups using STEP and the California Arithmetic Reasoning and Fundamentals tests but both gained on the average slightly more on the California test than did the similar group in the previous year learning conventional materials. Between classes having high ability students and those not having them there were no differences in achievement for the remaining eighty per cent except for the classes without high ability students on the California test where the difference was in their favour. "No matter which curriculum is used, homogeneous grouping does not benefit high ability students unless something special is done to take advantage of the grouping. Either the class should progress faster, or the course should be enriched, for example, with the SMSG supplementary materials."

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[39] School Mathematics Study Group, Reports on Student Achievement in SMSG Courses, Newsletter no. 10, November, 1961.

Experimental Studies of Attitude Toward Mathematics

The results of the foregoing studies of enrichment in mathematics along with grouping and acceleration as compared with enrichment have been determined on the basis of achievement. The objectives of a program of enrichment in mathematics should be realized also in terms of improved attitude. The studies which follow make some attempt at such evaluation.

Payne[40] reports a more enthusiastic attitude toward mathematics on the part of those participating in enrichment studies by following the twenty-four topics which he prepared for individual study. Content which was new and novel helped to motivate and there was more interest in mathematical ideas than in computation.

Albers and Seagoe[41] used an interest inventory of eight questions at the beginning and end of their experiment and found no significant changes in interest, but, for the experimentals, responses were consistent in showing a maintained or increased interest or more favourable attitude toward mathematics and science as compared with non-mathematics-science interests. For the control group responses were less consistent.

In the Cheltenham[42] experiment the teachers designed an attitude

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[40] Payne, op. cit., pp. 116-17.

[41] Albers and Seagoe, op. cit., p. 490.

[42] Passow, Goldberg, & Link, loc. cit.

test dealing with mathematics and society, the nature of mathematics, self appraisal of mathematics ability, and the school's effectiveness in teaching mathematics. After statistical tests the writers state there were significant differences on some of the categories between the number of "correct" or "positive" responses and those not "correct" favouring the acceleration group and the enrichment group that used the Illinois program.

Baker[43] surveyed 600 junior high schools in Michigan to determine the nature and extent of special provisions for the gifted in mathematics and science. The eighty-three per cent response indicated that a minority of the schools provided enrichment, acceleration, or a combination of the two. An opinionnaire of the principals', teachers', and students' perceptions of the effectiveness of a plan of accelerating eighth graders in eight schools into algebra showed no great differences between the three schools using one half year of acceleration and the five using a full year of acceleration. In five of nine questions the full year accelerating students did, however, show greater satisfaction toward their special programs and toward their work in general.

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[43] Russel R. Baker, "Program provisions in Michigan junior high schools for superior students in mathematics," The Mathematics Teacher, LV (November, 1962), pp. 556-59.

IV CONCLUSIONS

Where it is planned that extra attention be given to gifted students, the method of selecting these students should receive careful attention. While intelligence and related tests will identify superior mental ability, observation of certain traits, such as superior thinking ability and study habits along with an attitude of curiosity, is also important in their selection. More objective methods of detecting these traits need to be developed as teachers' observations are subject to error.

When enrichment has been used as a means of challenging the gifted in mathematics, teachers or local school systems usually have had to prepare their own materials. The topics are in keeping with the true nature of mathematics and go beyond the scope of the regular classroom studies. Often they are recreational. Various topics have been adapted for use at different grade levels. Topics first used as enrichment tend gradually to be absorbed into the regular mathematics course for all students. There is a great need for prepared enrichment materials ready for the teacher to use with gifted students in the regular classroom.

Most of the experiments in enrichment in mathematics are too small to be able to generalize conclusively (and many of the writers acknowledge this) but a few trends may be noted. Children gifted in mathematics are capable of benefitting from suitable special provision. At the junior high level they can accelerate as much as one year without

any harmful effects. Results of experimental work on enrichment are less clear than those on acceleration but enrichment can be profitable for brighter children provided it is sufficiently challenging. Grouping the gifted student separately from the less able in itself may not ensure greater progress of the gifted unless special provisions of enriched curriculum are also provided. It is possible that in mathematics improved attitude, greater appreciation and increased interest can result from enrichment studies.

One of the strongest arguments in favour of enrichment as compared with acceleration is stated by Dockrell.

> While acceleration does not seem to make for intellectual or social problems for the superior child, it is not immediately obvious why a routine Grade XI curriculum should be more stimulating than a routine Grade X curriculum. At the most, acceleration allows a youngster to save time by getting through school more quickly. It seems rather a pity however, if all the school can do for the gifted youngsters is to stand out of their way.[44]

Scheifele argues similarly in saying that acceleration does not ensure the fullest expression of the gifted child's powers nor meet all his needs.[45] Goldberg says there is no proof that an accelerated able group would do better than an equally able group which has had a year or two of stimulating advanced work.[46]

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[44] W. B. Dockrell, "Special Education for Gifted Children," The Canadian Educational and Research Digest, II (March, 1962), p. 45.

[45] Scheifele, op. cit., p. 44.

[46] Miriam L. Goldberg, "Research on the Talented," Teachers College Record, LX (December, 1958), p. 155.

CHAPTER III

THE CRITERIA

I A DESCRIPTION OF THE SCHOOL

While research indicates that each of the types of provision for gifted students has its advantages and disadvantages and that each can contribute to academic achievement,[1] the situation in the local school will in the end determine the nature of provision that can be made.

> The kinds of modifications needed in a particular school for the development of more adequate mathematics experiences for gifted students will be determined by the size of school, nature of professional staff, community setting, and similar factors.[2]

The School Organization

The enrichment topics presented in the fourth chapter of this thesis in line with the criteria that follow in the present chapter are designed for use in a small high school where special classes for the gifted are not possible. Specifically the investigator has in mind the school where she has been a staff member from 1956 to 1961, Woodstock

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[1] A. Harry Passow, "Talented Youth: Our Future Leaders" Teachers College Record, LVII (December, 1955), p. 166.

[2] R. J. Havighurst, E. Stivers & R. F. DeHaan, A Survey of the Education of Gifted Children, (Chicago: University of Chicago Press), p. 29, cited by A. Harry Passow & Deton J. Brooks Jr., New Developments in Secondary-School Mathematics, Bulletin of the National Association of Secondary-School Principals, 44 (May, 1959), p. 70.

School, located in the Himalayan mountains in India.

The copy of the school profile (see Appendix, Figure 5) provides a brief description of the nature of the school. A minor correction should be pointed out. The students are children of American missionaries predominantly and there are children of Canadian, British and other missionaries and of people attached to foreign embassies and foreign aid programs in India and nearby countries. Some Indian children are enrolled also.

Since the school is a residential school, students can easily participate in out-of-class activities such as music, drama, athletics, and various clubs. In fact the problem of a few students overloaded with activities is sometimes serious. Consequently there is a point system whereby each out-of-class activity has a point rating and the student must choose, with the help of his counsellor, activities totalling a certain maximum number of points depending upon his achievement. The academic school year extends from June through May with three months free for the long vacation from December through February. Climatic conditions of winter at a 7,000 foot altitude, a summer monsoon season, and the coming and going of people from schools in the U. S. A. and other countries make this the best possible arrangement. What Woodstock students learn in July, students in the U. S. A. will learn in September or October.

There is a cumulative record for each of the students, some of whom have ten of their twelve years of schooling at Woodstock School.

Achievement scores, aptitude and ability ratings, and personality inventories are recorded. The testing program of the school outside of teacher evaluation of pupil progress in class is indicated in Figure 6 of the Appendix. Results are used by the college guidance staff in advising students as to choice of college and field of study, and they are used in completing transcripts.

The Staff

Staff members come from the U. S. A., Canada, The United Kingdom, India, and sometimes Australia or continental Europe. Most are qualified teachers but some are college graduates who have not had formal teacher training. The ratio of staff members to students is about one to fifteen. Travel to and from India each year results in a fair turnover among staff.

The Students

The number of students and their ability are indicated on the profile (Appendix, Figure 5). In the National Merit Scholarship Qualifying Tests and the tests of the College Entrance Examination Board the range indicated is not a true picture for the entire senior class since it is the better students who write these tests. Concerning home background of the 155 enrolled in high school in the year 1962-63, 91 of 140 fathers have done graduate studies and for the mothers this number is 61. The general atmosphere of the school is one of high respect for superior achievement. The giving of annual awards in several academic

and other fields is one indication of this. On the whole, students are highly motivated and appreciate the importance of a good education.

The Mathematics Courses

Until 1963 there has been a three-five-four grade organization in Woodstock School which was then shifted to a six-six organization. Up to 1963 the mathematics classes in grades nine through twelve have used the textbooks:

Lennes, Maucker, & Kinsella, A First Course in Algebra, Macmillan & Co., 1957. (Grade nine).

Lennes, Maucker, & Kinsella, A Second Course in Algebra, Macmillan & Co., 1957. (Grade eleven).

Welchons, Krickenberger, & Pearson, Plane Geometry, Ginn & Co., 1958. (Grade ten).

Butler & Wren, Trigonometry for Secondary Schools, D. C. Heath, 1948. (Grade twelve).

Seymour & Smith, Solid Geometry, Macmillan & Co., 1949. (Grade twelve).

A changeover to the new mathematics programs prepared by the School Mathematics Study Group is going on at the present time beginning in grade nine in June, 1963.

For graduation the school requires two years of mathematics study but a majority of the students go beyond this minimum.

During the past five years, when a qualified mathematics teacher has been available, the Advanced Placement Program in Mathematics of the College Entrance Examination Board has been in effect. The program does not commence until the student's eleventh year when he takes an

extra mathematics course, trigonometry, for one semester. Solid geometry is studied for the first semester of grade twelve and the course for the second semester is calculus, a topic of the Advanced Placement Program. An average of four or five students follow this program.

Equipment, Library and Textbooks

The school has a library with appropriate study space and a librarian. There are around 10,000 volumes. Since new purchases are made annually for each department including mathematics, most probably the New Mathematical Library, containing volumes of the Monograph Project of the School Mathematics Study Group, will be added to the library as the SMSG courses are introduced into the classrooms. For many of the courses taught in the high school there are supplementary references or textbooks in the library and there are subscriptions to periodicals such as the Scientific American.

The textbooks for all courses are purchased from the U. S. A. and loaned to the students.

There is also a well equipped industrial arts room in the school.

II COMMENTS REGARDING CRITERIA

Descriptions of enrichment or other plans for the gifted in mathematics usually include criteria for the students, criteria for the studies, and sometimes criteria in general.

Specific criteria for the students who are preparing to study Advanced Placement Mathematics are summarized in Kieffer's article.[3]

1. Mental ability test--I. Q. 120 or above.
2. Mathematics-- PR ninety or above.
3. First semester (ninth grade). School grade point average--3.5 or above in mathematics, (four points - A).
4. First semester (ninth grade). Total grade point average--3.0 or above in academic subjects.
5. Teacher recommendation.

In his report of provisions for gifted students in certain Minneapolis schools, Jackson described the superior pupil as one to whom most of the following statements would apply:

> He obtains an I. Q. score of 120 or above on a group test of mental ability.
> He attains a percentile rank of ninety or better on standard mathematics achievement tests.
> He has A marks in previously taken mathematics courses.
> He shows strong interest in learning mathematics.
> He has good work and study habits.[4]

For the enrichment studies one restriction stated repeatedly in the literature is that they ought not to consist of larger quantities of work of the same type. Payne is in agreement here and says more positively that enrichment "should entail content and activities that require a high order of logical thinking and creative imagination; it

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[3] Mildred, Kieffer, "Meeting the Needs of Cincinnati's Gifted Pupils in Mathematics," New Developments in Secondary-School Mathematics, Bulletin of the National Association of Secondary-School Principals, 247 (May, 1959), p. 90.

[4] Harvey O. Jackson, "The superior pupil in mathematics," The Mathematics Teacher LII (March, 1959), p. 202.

should deepen and strengthen understandings."[5]

In setting up their studies Albers and Seagoe attempted to abide by the following six criteria:

1. Is the material interesting?
2. Is it material which is not usually presented to these children in organized form in any other course?
3. Is it difficult enough to offer a challenge, but not so difficult as to cause discouragement?
4. Does it contribute to a cultural background?
5. Is it presented in such a way as to develop habits of independence in study?
6. Does it utilize and develop the superior ability to see relationships to organize data, and to summarize findings?

Dickie describes the building of a supplementary program in junior-high school mathematics to be built around certain tenets some of which are stated here.

3. A supplementary program in mathematics should be workable without causing administrative problems such as regrouping or rescheduling.
4. Program should present a minimum of social and emotional problems among students such as feelings of superiority, inferiority, resentment.
5. Program should take up a minimum of class time.
6. Program should be self-sustaining; i.e. a student, if he desires, should be able to continue working without having to depend upon the teacher to carry him from stage to stage of the program.
7. Program should be set up to allow students to proceed at individual rates.
8. Program should be optional. Carrying force should be provided

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[5] Joseph N. Payne, "Self-instructive enrichment topics for bright pupils in high school algebra," The Mathematics Teacher, LI (February, 1958), p. 114.

[6] Mary E. Albers, & May V. Seagoe, "Enrichment for Superior Students in Algebra Classes," Journal of Educational Research, XL (March, 1947), p. 483.

primarily by the intrinsic nature of the program and secondarily by the motivating powers of the teacher.

9. Program should be almost entirely above and beyond the regular classroom work.
10. A large variety of cumulative and non cumulative topics should be available to appeal to individual differences in preference to quality and quantity of subject matter.
12. Program may offer contact with some modern ideas in mathematics not normally found in the junior high mathematics class.[7]

From the point of view of the studies only, two criteria probably stand out as the most important ones for enrichment. In Wirszup's article they are contained in his definition, while in the Report of the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics they are stated as requirements.

The first criterion is subjective. A program of enrichment for the gifted should "provide real challenge to excellent accomplishment."[8] It should provide mathematical experience that is "an awakening and quickening,--quickening the student's interest in mathematics and awakening his sense of mathematical power."[9]

The second criterion is objective. A program of enrichment for the gifted should "provide basic information essential to top level

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[7] Paul Dickie, "A supplementary program in junior-high-school mathematics," The Mathematics Teacher, LV (January, 1962), p. 57.

[8] The Secondary Mathematics Curriculum, Report of the Secondary School Committee (Washington, D.C: National Council of Teachers of Mathematics, May, 1959), p. 411.

[9] Izaak Wirszup, "Some remarks on enrichment," The Mathematics Teacher, XLIX (November, 1956), pp. 519-20.

performance in mathematics, both for the present and for the future."[10] It should make provision for "broadening the mathematics to which the student is exposed with reference to other significant mathematics."[11]

III THE CRITERIA FOR ENRICHMENT IN MATHEMATICS AT WOODSTOCK SCHOOL

An enrichment program in mathematics in the early years of high school at Woodstock School would involve certain criteria to be met by the school, the staff, the students and the studies.

The Criteria for the School

The provision of library facilities including regular purchasing of suitable supplementary mathematics textbooks, materials and periodicals should continue. The purchases ought to include some of the recently available enrichment materials such as the series *Exploring Mathematics on Your Own* by Donovan A. Johnson and William H. Glenn, published by Webster Publishing Company and also the volumes of the New Mathematical Library mentioned early in the present chapter. Some programmed material should be obtained. As example of this is a unit on *Groups and Fields* by Boyd, Moore and Smith, McGraw Hill Book Co., 1963. There should be a subscription to *The Mathematics Student Journal*. A few books such as the following on history of mathematics and on modern

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[10] *The Secondary Mathematics Curriculum*, *loc. cit.*

[11] Wirszup, *loc. cit.*

mathematics would be useful.

Irving Adler, The New Mathematics, John Day Publishing Co., New York. 1958.

Richard Courant & Herbert Robbins, What is Mathematics?, Oxford University Press, New York, 1941.

Dantzig, Tobias, Number The Language of Science, Macmillan Co., New York, 1954.

Kemeny, John George, Snell, J. Laurie, & Thompson, Gerald L., Introduction to Finite Mathematics, Prentice-Hall, Englewood Cliffs, 1957.

National Council of Teachers of Mathematics, Enrichment Mathematics for High School, Twenty-Eighth Yearbook, Washington, D. C., 1963.

_______, Enrichment Mathematics for the Grades, Twenty-Seventh Yearbook, Washington, D. C., 1963.

Sawyer, W. W., Prelude to Mathematics, Penguin Books, Baltimore, Md., 1955.

Reid, Constance, From Zero to Infinity, Routledge & Kegan Paul, London, 1956.

Most of these books may be found in a very complete list of books recommended for the library of the high school in the Twenty-Eighth Yearbook of the National Council of Teachers of Mathematics.

2. There should be an attitude of encouragement on the part of the school administration, especially the high school supervisor, so that the teacher has some time for the mechanics of operating an enrichment program and so that some time outside of class hours is available for meeting with students, say as a mathematics club.

The Criteria for the Staff

1. The mathematics teacher ought to have an interest and

enthusiasm which will inspire the student to enquire, investigate, and search in order to find out what he does not know.

2. The mathematics teacher should have adequate but not necessarily advanced training. Rourke summarizes the minimum essentials in the professional education of a prospective teacher of mathematics. In the junior high school these are: four years of high-school mathematics, eighteen hours of college mathematics (Calculus and Analytical Geometry; Algebra and Number Theory; Geometry), and three or four credit hours in Mathematics Education. For the senior high school these are the same except that there should be thirty hours of college mathematics (Calculus and Analytical Geometry; Analysis; Abstract Algebra; Geometry; Statistics; Logic).[12]

3. The enrichment program ought not to involve an excess of the teacher's extra time.

The Criteria for the Students

1. The student should have a minimum I. Q. rating of 125. The I. Q. equivalent on the Lorge-Thorndike Intelligence Test 4V (mean-100, s. d.-16) should be available for most of the students. An alternative to the I. Q. rating would be a minimum percentile rank of ninety on the Numerical Ability and Space Relations portions of the Differential

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[12] R. E. K. Rourke, "The Commission of Mathematics of the CEEB and Teacher Education," New Developments in Secondary-School Mathematics, The Bulletin of the National Association of Secondary-School Principals, 44 (May, 1959), p. 178.

Aptitude Tests (or an average of ninety for these two ratings). This battery of tests is given all students of grade nine very shortly after the beginning of the first semester.[13]

2. On the Arithmetic Skills portion of the Iowa Tests of Basic Skills a percentile rank of ninety-four or more should be required. The class median for Woodstock students tends to be high on these tests and the school is at present switching to Special Percentile Norms for Socio-Economically Select Districts prepared for use with the Iowa Tests. With these norms the minimum requirement will be considerably lower.

3. The student's average grade in mathematics during the previous year should be 3 2/3 or above, (A or A-), on a four point scale, (A = 4).

4. The over all previous academic average should be not lower than 3.0, (B).

5. There should be recommendations from three previous teachers (or the high school supervisor and two teachers) regarding characteristics indicating evidence of satisfactory study habits, initiative, ability to think and deal with the abstract, and an attitude of curiosity.

6. A point rating for mathematics enrichment studies as a part of the point rating of all out-of-class activities so that the total load for the individual is not so great as to cause emotional upset or strain. Some students would have one hour per week available for

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[13] see Figure 6 of the Appendix.

enrichment studies and a few may have four hours.

7. Parental consent.

In each class of thirty to thirty-five students there would be approximately eight or fewer who would qualify according to the above requirements. The student could start at the beginning of the school year or later. Participating in enrichment work would be treated as a privilege that may be removed if the student falls back seriously in any other regular work as a result of putting too much effort into the special study.

Sometimes a new student coming in may not have any rating that would suffice for the first two criteria listed. The staff would have to depend upon the rating on the ACE Psychological test and judge more on the basis of the third and fifth criteria. Most incoming students do, however, have a transcript with some of this information.

A check list containing items similar to those mentioned in chapter II in connection with identification of the gifted[14] would help in securing teachers' recommendations for criterion five. Norton suggests a list containing fifteen items dealing with mental traits of the gifted pupil but it provides for a check only and not for any indication of the degree to which the student possesses some trait.[15]

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[14] see pp. 13-15.

[15] Monte S. Norton, "What are Some of the Important Factors to Consider in a Program of Identifying the Gifted Pupil in Mathematics and Science," School Science and Mathematics, 57 (February, 1957), p. 107.

The student who does not qualify on criterion four but who gives indication of a very high ability in mathematics may profit from enrichment work. Here again criterion five would be of greater importance in making a decision.

The Criteria for the Studies

1. The studies should be advanced beyond the work being done in the regular classroom but not seriously overlapping the content of other mathematics courses. The studies may but need not necessarily arise directly from the topics being covered in the regular class.

2. The studies should be stimulating, challenging, and satisfying; where possible there should be opportunity for the student to arrive at a generalization or to discover for himself. Studies meeting these requirements should provide intrinsic motivation and they should develop appreciation for mathematics; the student should enjoy the content of the topics and their development.

3. The studies should be self administering, prepared so that the individual student can proceed with very little teacher guidance; there should be no special classes to prepare the students for the work, only private counselling as necessary. The student should proceed at his own rate and do as much as he is able during the year.

4. The student's work is not to be graded and no credit is earned. Extra credit in the form of some recognition of satisfactory completion of specific topics should be noted on the cumulative record.

A means of assessing quickly the student's work may be through a simple self-administered test, an oral report, or a written report.

5. The content of the studies should be mathematical, that is, topics that can be followed if one has some mathematical background; the content should contribute toward the pupil's total grasp or understanding of mathematics.

6. The topics should be community orientated in that there should be recognition or emphasis upon mathematics and the Indian (Hindu) setting and upon contributions to mathematics from this part of the world.

SUMMARY

For Woodstock School for foreign children in India, the criteria for the school, the staff, the students, and the studies are outlined in the present chapter for the purpose of introducing a program of enrichment in mathematics in grades nine and ten for gifted students. The students, English speaking foreign children living in India and nearby countries, are, on the average, brighter than students in the usual heterogeneous classroom. The criteria as set up for the students will probably permit up to one-fourth of the number in a mathematics class to qualify as gifted students who may participate in the enrichment studies. The enrichment work is of the type that the pupil follows on his own outside of class.

The six units of study presented in Chapter IV are prepared to comply with the criteria for the studies as stated in the present chapter.

CHAPTER IV

THE ENRICHMENT TOPICS

INTRODUCTION

The present chapter contains six units prepared for individual study by students in grades nine and ten. Specific topics are not restricted to either one of the two grades though in some instances the nature of the topic is such that it is more suitable for one of them. The study of logic and finite geometry is intended for grade ten and, except for the last exercise, the topic on Pythagorean triples is more suitable for grade nine. Space for the student's work would normally be provided in topics of this nature but as set up for this chapter it is omitted in most of them. Some of the topics require a certain amount of outside reading but others, such as the one on logic, are complete in the form given here. References are given at the end of each topic and special items taken directly from specific sources are acknowledged in footnotes.

Certain assumptions go along with particular topics. For example it is assumed the student has had some practice in modular arithmetic at some time before high school in order to follow the topic on finite arithmetic. It is obviously assumed that the student is slightly familiar with the country of India though this is not absolutely

necessary even for the topic dealing with India and its contributions.

Generally each topic meets the requirements of all of the criteria for the studies as stated in Chapter III but in some instances certain criteria are more prominent.[1] The topic concerning India's contribution to mathematics, for example, fulfils criterion six primarily.

The list of topics is as follows:

The Pythagorean Theorem and Pythagorean Triples

A Consideration of π

India's Contribution to Mathematics

Finite Geometry and Non-Euclidean Geometry

Logic

Finite Arithmetic and Congruences

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[1] see p. 59.

I THE PYTHAGOREAN THEOREM AND PYTHAGOREAN TRIPLES

Pythagoras, whose name is connected with the proof of the theorem about the right-angled triangle, was a Greek mathematician who lived before 500 B. C. He established a society or brotherhood at Crotona in southern Italy. The Pythagoreans, as members of the society were called, were interested in number and number patterns such as the geometric pattern that develops when one adds consecutive odd numbers.

1	$= 1$	$1 = 1^2$	*
$1 + 3$	$= 4$	$4 = 2^2$	* * * *
$1 + 3 + 5$	$= 9$	$9 = 3^2$	* * * * * * * * *
$1 + 3 + 5 + 7$	$= 16$	$16 = 4^2$	

Exercise 1

(a) Do you recall a way of finding the sum of the first six consecutive odd integers, a short cut method, so that you need not add the numbers one at a time? Do this for the series that follows where dots are used to indicate that all the missing consecutive odd integers are included though not written out in full.

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 11 =$

Since the numbers are evenly spaced you can find the average or mean of the first and last and multiply by the total number of terms of the series.

(b) By the same method find the following sums:

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 23 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 25 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 47 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 49 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 79 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 81 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 119 =$

$1 + 3 + 5 + \ldots\ldots\ldots\ldots + 121 =$

Pythagorean Triples

Any three whole numbers in place of a, b, and c which satisfy the relation $a^2 + b^2 = c^2$ are said to be Pythagorean triples or numbers.

Exercise 2

(a) Refer to part (b) of exercise 1 to find sets of Pythagorean triples. For example:

$144 + 25 = 169$

$12^2 + 5^2 = 13^2$

$a = 12,\ b = 5,\ c = 13$

(b) The numbers 12, 5, and 13 form a set of primitive Pythagorean

triples because they are relatively prime; they have no common factor other than 1. Find a set of Pythagorean triples that are not primitive.

(c) There are formulas that can be used to obtain sets of Pythagorean triples when certain instructions are followed. Note the two sets of formulas.

(i) $a = n, \quad b = \frac{n^2 - 4}{4}, \quad c = \frac{n^2 + 4}{4}$

where n is a positive even integral number.

(ii) $a = m, \quad b = \frac{1}{2}(m^2 - 1), \quad c = \frac{1}{2}(m^2 + 1)$

where m is a positive odd integral number.

Use these formulas to obtain at least two sets of Pythagorean triples not contained previously in this unit.

n	$n^2 - 4$	$n^2 + 4$	a	b	c

m	$m^2 - 1$	$m^2 + 1$	a	b	c

(d) What happens if n is odd or if m is even? Do you get replacements for a, b, and c that satisfy $a^2 + b^2 = c^2$?

(e) It is stated that a study of the series involving consecutive odd numbers led the Pythagoreans to look for number replacements that would satisfy not only the relation $a^2 + b^2 = c^2$ but also other relations such as $a^2 + b^2 + c^2 = r^2$. Can you suggest other relations for which an attempt to find replacements might be interesting?

(f) Changing the exponent could give the relation $a^3 + b^3 = c^3$. In fact it is possible to think of a more general relation, $a^n + b^n = c^n$ where $n \geqq 3$. Consult one of the books listed below or a history of mathematics and write a paragraph on the relation just stated. Mention the work of Fermat in this regard; give a statement of Fermat's last theorem and comment on the proof of it.

Invitation to Mathematics, Donovan Johnson & William Glenn.

From Zero to Infinity, Constance Reid.

Number the Language of Science, Tobias Dantzig.

(g) Concerning the sums of cubes, G. H. Hardy tells a story about the Indian mathematician Ramanujan. He went to visit Ramanujan and commented that the number on his taxicab, 1729, was a rather dull number. The latter replied, "No, it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." Read this story on page 94 of The Lore of Large Numbers by Philip Davis and complete number 19 of Problem set 20 as you find it there.

Proofs of the Pythagorean Theorem

There exist many proofs of the Pythagorean theorem other than the

one given by Pythagoras. There is one attributed to the Hindus in India and another to President Garfield.

Exercise 3

(a) Find the other proofs just mentioned in your textbook and write these proofs for the relation $a^2 + b^2 = c^2$ where a, b, and c represent the length of the two legs and the hypotenuse respectively of a right angled triangle. These are algebraic proofs.

Plane Geometry, Welchons, Krickenberger & Pearson, p. 417.

References

Dantzig, Tobias, Number the Language of Science, The Macmillan Co., 1954, pp. 52-56, 285-90.

Davis, Philip , The Lore of Large Numbers, New Mathematical Library, The L. W. Singer Company, Random House, 1961, p. 94.

Johnson, Donovan, & Glenn, William, Invitation to Mathematics, Exploring Mathematics on Your Own, Webster Publishing Co., 1961, p. 52.

______, Number Patterns, Exploring Mathematics on Your Own, Webster Publishing Co., 1961, pp. 25-30.

______, The Pythagorean Theorem, Exploring Mathematics on Your Own, Webster Publishing Co., 1960, pp. 6-16.

Reid, Constance, From Zero to Infinity, Routledge & Kegan Paul, 1956, pp. 65-68.

Welchons, A. M., Krickenberger, W. R., & Pearson, H. R., Plane Geometry, Ginn and Company, 1958, p. 417.

II A CONSIDERATION OF π

Introduction

Sometimes when you and a classmate have checked problems in determining the area of a circle you may have been annoyed to find that while neither of you had any error your answers did not agree. One of you used the value for π as 3.14 and the other used the value 22/7. The first of these is a little less than π and the second is a little larger, that is, $3.14 < \pi < 22/7$.

Does π have an exact value? What is the definition of π? You know that it can be defined as the ratio of the circumference of a circle to the length of its diameter. Expressed in symbols this says $\pi = C/d$ where C represents the circumference of a circle and d its diameter. As to an exact value, the remainder of this unit is directed toward trying to find an answer to this question.

Early Historical Note on π

The Babylonians and Hebrews used 3 as a value for π. Check for yourself these references in the Bible: I Kings vii 23 and II Chronicles iv 2. The Egyptians used the value $\frac{256}{81}$ or 3.16050... . One early method used by several people to obtain an approximate value for π was that of comparing the perimeters of two regular polygons, one inscribed in a circle, the other circumscribing it. The circumference of the circle would be between the two values for the perimeters of the polygons and from this a value of π would be found. In Plane Geometry

by Welchons, Krickenberger and Pearson read pages 469-70 to get a better idea of how this is done.

Exercise 1

Read the first ten pages of Part II of The Lore of Large Numbers by Philip Davis for a further description of this method of evaluating π and other methods.

(a) Archimedes used polygons of how many sides in trying to find a value for π?

(b) State his result as an inequality.

(c) Give the names and findings of at least three other men who attempted to evaluate π by the same method.

(d) Do the first three problems of Problem Set 17 on page 65.

Non Repeating Decimals

It was the hope of those who attempted to evaluate π that they would find its value to be a fraction. It is now known that π cannot be expressed as a fraction. To consider this more closely, two ideas, one the converse of the other, are explored by Davis in The Lore of Large Numbers, pages 59-60 and 80, and by Niven in Numbers: Rational and Irrational, pages 23-26 and 30-37. These are:

a) Every fraction when expressed as a decimal either "comes out even," or it is cyclic, that is, a repeating decimal.

$1/8 = 0.125$ $\qquad\qquad$ $1/3 = 0.3333...$

b) Every repeating decimal must be a fraction.

Let r be such a repeating decimal.

$$r = 0.4062762762\ldots$$

$$1000r = 406.2762762762\ldots$$

$$999r = 405.87$$

$$r = \frac{405.87}{999} = \frac{40587}{99900}$$

Exercise 2

(a) The Lore of Large Numbers, Problem Set 19, page 82, 1 & 2.

(b) Numbers: Rational and Irrational, Problem Set 6, page 26.

Problem Set 8, page 34.

More Ways of Finding a Value for π

Mathematicians eventually proved that π was not a fraction or repeating decimal. Further they learned that it is not related to the circle only but also to other closed curved figures. Viete (1540-1603), a French Lawyer, was the first to find a way of calculating the value of π that did not involve finding the circumference of a circle. In time it was learned that π is not confined to geometry.

Exercise 3

(a) Indicate the contribution and the year associated with it for these people who added to the growing amount of knowledge about π.

Lambert

Liouville

Lindemann

(b) What is the meaning of transcendental?

Periods of Renewed Interest in the Value of π

If you have read nearly all of Part II of The Lore of Large Numbers you have noticed that the interest in computing the value of π has thrived during some periods and at other times it has been dormant. One person wrote in 1911 that there have been three well-defined periods in the story of the effort to determine π. The first extended from earliest times up to about 1650 A. D. There was the search for a way to find a square equal in area to a given circle and there were the efforts to find an approximate value of π by geometric means. The second, about a century in length, begins at the time of the invention of the differential and integral calculus and ends about 1766 with the work of Lambert. Infinite series and products were used to approximate the value of π. During the third period, extending from the middle of the eighteenth century to our day, people were trying to find out about the nature of π, whether it is rational or not, whether it is algebraic or transcendental. If the writer were writing in 1963 he might cut off the third period at the beginning of our century and speak of the present day as a fourth period.

Exercise 4

(a) Note the list of infinite series relating to π as in Figure 25 on page 16 of The Lore of Large Numbers. Davis says that these series indicate that π is a property of all mathematics and not of

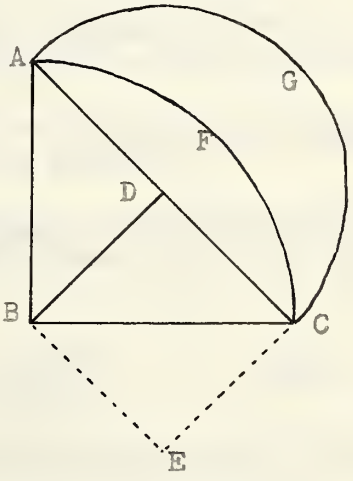
geometry alone. What simple explanation of an infinite series is given here?

(b) You know of the ancient problem of "squaring the circle," the attempt to construct with straight edge and compass a square equal in area to a given circle. This would involve finding a way to construct a line equal in length to the circumference of a circle and to do this a value of π was needed since $C = \pi d$. What was the effect of the proof of Lindemann on this quest in 1882?

(c) Suppose we speak of the time beginning in 1940 as a fourth period in the search for a value for π. Describe this period.

(d) Calculate π experimentally according to the instructions given in Number Patterns by Donovan Johnson and William Glenn, Exercise Set 12, page 39, number 2. This method is related to the idea of probability. Line a page of paper in order to tabulate your counts of sticks.

(e) In the diagram on this page B is the center of arc AFC and D is the center of arc AGC. Angle ABC is a right angle and BD is perpendicular to AC. Prove the crescent AFCG equal in area to square BDCE.



The problem of squaring a crescent is one which can be done. The first one to do it was

Hippocrates. Try the proof yourself. If you need help go back to the section you read in The Lore of Large Numbers.

(f) A value of π was known in India as early as 500 A. D. Aryabhatta gave the value $\frac{62832}{20000}$ which is equal to 3.1416. Brahmagupta, more than a century later, gave the square root of ten as the value. What value did Srinavasa Ramanujan give in 1914? (Lore of Large Numbers, page 65).

(g) Make a chart suitable for the bulletin board showing the outstanding events in the story of π. Organize it as you wish. There may be columns for the date, the country, the person who made some contribution, and the contribution, or you may prefer to use sections for different periods in the story of the search for π. Include suitable sketches if you care to do so.

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III INDIA'S CONTRIBUTION TO MATHEMATICS

> It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.
>
> --Laplace[2]

In this unit we shall consider India, some of her great mathematicians and their work, and the contributions of India to mathematics in general. We shall see some mathematical procedures a little different from those we know. There is outside reading in this unit and a few problems later, some not very simple. Part of what you read may be slightly vague because the early records are often obscure and the writings of mathematicians usually included philosophy and religion along with mathematics. As you might expect problems were written in verse since they were often used for entertainment. Some sample problems are included but you do not have to solve all of them.

The Hindu-Arabic Numerals

When using Roman Numerals, if you had to find the sum of LXVIII and XXXIII or the product of CXLII and LXXXIV, you would rewrite the

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[2] Tobias Dantzig, Number the Language of Science, (New York: The Macmillan Co., 1954), p. 19.

problems as 68 + 33 and 142 x 84 and then compute by the method that is familiar to you. There was a time in Europe until about the end of the Dark Ages when the people did not have our present Hindu-arabic numerals and computation was carried out usually on an abacus.

Exercise 1

Consult a book or article on the history of mathematics in order to learn how we obtained our numerals from India. Write a paragraph on the Hindu-arabic numerals including most of these items:

The pillars of King Asoka
The nine Hindu numerals and positional value
The accidental discovery of zero (Sanskrit sunya)
The names of words for zero,--sunya, sifr, cipher
The transfer of the numerals to Arabia and to Europe
The battle between the abacists and algorists

You will find these books helpful:

Number the Language of Science, Tobias Dantzig
From Zero to Infinity, Constance Reid
An encycloepedia

A Series of Problems[3]

Here are several problems from the writings of Hindu mathema-

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[3] Most of the problems given here and others like them may be found in Howard Eves, An Introduction to the History of Mathematics, (New York: Rinehart and Company, Inc., 1956), pp. 197-200.

ticians. It is intended primarily that you enjoy reading them but you may solve as many of them as you wish or all if you are able.

(a) Out of a heap of pure lotus flowers, a third part, a fifth and a sixth were offered respectively to the gods, Siva, Vishnu, and then Sun; a quarter of the original heap was presented to Bhavani. The remaining six lotuses were given to the venerable preceptor. Tell me quickly the whole number of lotus flowers.

(b) A necklace was broken during an amorous struggle. One-third of the pearls fell to the ground, one-fifth stayed on the couch, one-sixth was found by the girl, and one-tenth recovered by her lover; six pearls remained on the string. Say of how many pearls the necklace was composed.

(c) The mixed price of 9 citrons and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and of a wood apple here, having distinctly separated those prices well.

(d) A powerful unvanquished excellent black snake which is 80 angulas in length, enters into a hole at the rate of $7\frac{1}{2}$ angulas in 5/14 of a day, and in the course of $\frac{1}{4}$ of a day its tail grows 11/4 of an angula. O ornament of arithmeticians, tell me by what time this serpent enters fully into the hole.

(e) A merchant pays a duty on certain goods at three different places. At the first he gives 1/3 of the goods, at the second $\frac{1}{4}$ (of the remainder), and at the third 1/5 (of the remainder). The total duty is

24. What was the original amount of goods?

(f) How many are the variations in the form of the god Sambu (Siva) by the exchange of his ten attributes held reciprocally in his several hands: namely, the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the arrow, the bow; as those of Hari by the exchange of the mace, the discus, the lotus, and the conch?

(g) Arjuna, exasperated in combat, shot a quiver of arrows to slay Carna. With half his arrows he parried those of his antagonist; with four times the square root of the quiverful he killed his horse; with six arrows he slew Salya (Carna's charioteer); with three he demolished the umbrella, standard, and bow; and with one he cut off the head of the foe. How many were the arrows which Arjuna let fly?

Algebra in India

Algebra as well as our number system originated in India but it reached Europe too late to be of influence. This algebra did not have symbols. Historians speak of rhetorical, syncopated and symbolic algebra. The first expresses a problem in words and uses no symbols. In syncopated algebra words that are used frequently are abbreviated. In India multiplication was indicated by writing bha (bhavita, "the product") after the factors, division by writing the divisor beneath the dividend, square root as ka (karana, "irrational") before the quantity. Yā (yāvattāvat, "so much as") represented the unknown, rū (rūpa, "the

absolute number") was the known integer. Kā (kālaka, "black") and abbreviations of names for other colours represented additional unknowns.[4]

Thus $8xy + \sqrt{10} - 7$ would look like this

yā kā 8 bha ka rū $\dot{7}$.

The dot over 7 represented subtraction. Juxtaposition indicated addition.

The Hindus knew about negative and irrational numbers, quadratic equations and their solution, compound interest and discount. They did a lot of solving of problems and they knew how to solve indeterminate equations. In geometry they did very little though they had some accurate values for π and knew a proof of the Pythagorean theorem. Unlike the Greeks they were not concerned about proof and clear logical reasoning; their work was more empirical, and while some was good mathematics, some was not. The Greeks developed mathematics for its own sake but astronomy dominated mathematics in India. In computation the Indians were excellent.[5]

Exercise 2

Again consult the encycloepedia or a history of mathematics to find what you can about India's early mathematicians from about 500 A. D. and on. Complete a chart similar to the one which follows or

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[4] Eves, op. cit., p. 185.

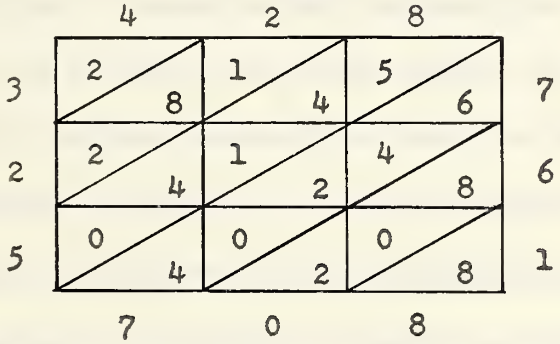
[5] Ibid., pp. 184-89.

enlarge on this if possible.

	Time	Place	Names of writings (and translation)	Contributions
The Aryabhatas				
Brahmagupta				
Mahavira				
Bhaskara				

Exercise 3

(a) Very likely you recognize the following and if you don't you should be able to find someone who does. In some village schools you can find this in practice today.



(b) The rule of three[6] stated by Brahmagupta says: "In the rule of three Argument, Fruit, and Requisition are the names of the terms. The first and last terms must be similar. Requisition multiplied by Fruit, and divided by Argument, is the Produce." Now look at a problem. If two and a half palas of saffron are purchased for three sevenths of a niska, how many palas will be purchased for nine niskas? 3/7 and 9

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[6] *Ibid.*, p. 192.

which are of the same denomination are the Argument and the Requisition and 5/2 is the Fruit. The Produce then is obtained in this way:

$$(9)(5/2)\ /\ (3/7) = 52\tfrac{1}{2}$$

Set up this problem by the method you would use.

(c) Two identities given by Bhaskara are:

$$\sqrt{a \pm \sqrt{b}} = \sqrt{(a + \sqrt{a^2 - b})/2} \pm \sqrt{(a - \sqrt{a^2 - b})/2}$$

Use these to simplify the expression $\sqrt{17 + \sqrt{240}}$

(d) Brahmagupta gave the formula

$K^2 = (s - a)(s - b)(s - c)(s - d)$ for the area of K a cyclic quadrilateral of sides a, b, c, d and semi-perimeter s. Show that the area of a quadrilateral possessing both an inscribed and circumscribed circle is equal to the square root of the product of its sides.

(e) Find the smallest permissible answer to the following indeterminate problem of Mahavira. "Into the bright and refreshing outskirts of a forest, which were full of numerous trees with their branches bent down with the weight of flowers and fruits, trees such as jambu trees, lime trees, plantains, areca palms, jack trees, date palms, hintala trees, palmyras, punnaga trees, and mango trees--outskirts, the various quarters whereof were filled with the many sounds of crowds of parrots and cuckoos found near springs containing lotuses with bees roaming about them--into such forest outskirts a number of weary travelers entered with joy. There were 63 numerically equal heaps of plantain fruits put together and combined with 7 more of these same

fruits, and these were equally distributed among 23 travelers so as to have no remainder. You tell me now the numerical measure of a heap of plantains."[7]

Here is another of the type of problems that interested the Indians. You may sometime learn a mathematical procedure for attacking such a problem. Meanwhile see what you can attempt to do with it.

References

Dantzig, Tobias, Number the Language of Science, The Macmillan Co., 1954, Chapter Two.

Eves, Howard, An Introduction to the History of Mathematics, Rinehart and Company, Inc., 1956, Chapter Seven.

Reid, Constance, From Zero to Infinity, Routledge & Kegan Paul, 1956, pp. 4 - 7, 117.

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[7] *Ibid.*, p. 200.

IV FINITE GEOMETRY AND NON-EUCLIDEAN GEOMETRY

Introduction

In your geometry course you discussed the fact that the study was to be developed in a way that was logical. You learned that it was necessary to begin with certain undefined terms and postulates, that is, statements that were accepted without proof. As your study proceeded you defined new terms and you proved theorems by logical reasoning.

Where do we first get the postulates? Are they true statements? Sometimes it is stated that they are self-evident because they agree with our experience of the world around us. Therefore it is argued that they are true. But it is not necessary that they be in agreement with our physical surroundings as we experience them. All that we can say is that postulates are statements that are assumed to be true. What is important is that from a set of postulates we be able to obtain theorems which may be proved to be true (valid is a more correct word) according to the rules of logic.

One of the postulates of geometry states that on a given point not on a given line there exists one and only one line parallel to the given line. This postulate may be replaced by another one to get another system of geometry about which you will read more at the end of this unit. Taking a set of postulates and omitting one or replacing it by another postulate is a usual way of obtaining a different set of postulates from which a new mathematical system may be deduced.

While we do not establish the truth of a set of postulates this does not mean one is free to choose postulates at random. If it is possible to prove a theorem both true and false from the same set of postulates, that set of postulates is inconsistent and of no value. It is necessary that a set of postulates be consistent. Further, it hardly seems sensible to accept as a postulate a statement that could be proved from the remaining postulates of a set. It is desirable that each postulate be independent of the others.

The small geometric systems that you will first consider in this unit may be called finite geometries since they contain only a finite number of points.

Finite Geometry

The following is a finite geometry in a plane, S. There are five postulates. The words "point," "line," and "plane" are undefined and the relations "point in common" and "point on a line" are also undefined. Postulates and theorems will be identified by the symbols P-1, P-2, etc., and T-1, T-2.

P-1 Each pair of lines has at least one point in common.

P-2 Each pair of lines has not more than one point in common.

P-3 Every point in S is on at least two lines.

P-4 Every point in S is on not more than two lines.

P-5 The total number of lines in S is four.

T-1 Each pair of lines in S has exactly one point in common.

T-2 Each point in S is on exactly two lines.

You can see that T-1 comes from P-1 and P-2. Every two lines must meet in one and not more than one point. There can be nothing similar to the representation of Figure 1 in this geometry where l_1 and l_2 are two distinct lines and P_1 and P_2 two distinct points.

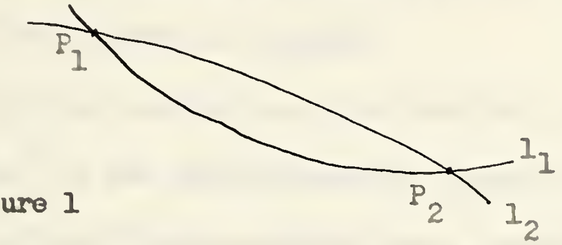

Figure 1

Similarly, according to T-2, there cannot be a representation like that of Figure 2 with only one line on a point.

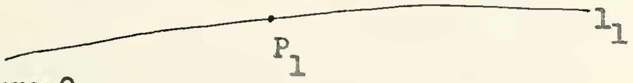

Figure 2

We see that exactly six points are necessary in order that every set of two lines may have one point in common. Let these be $A_{(1,2)}$, $B_{(1,3)}$, $C_{(1,4)}$, $D_{(2,3)}$, $E_{(2,4)}$, and $F_{(3,4)}$. Let the lines be L_1, L_2, L_3, and L_4. L_1 and L_2 have the point $A_{(1,2)}$ in common. The representation of Figure 3 satisfies the set of postulates P-1 to P-5.

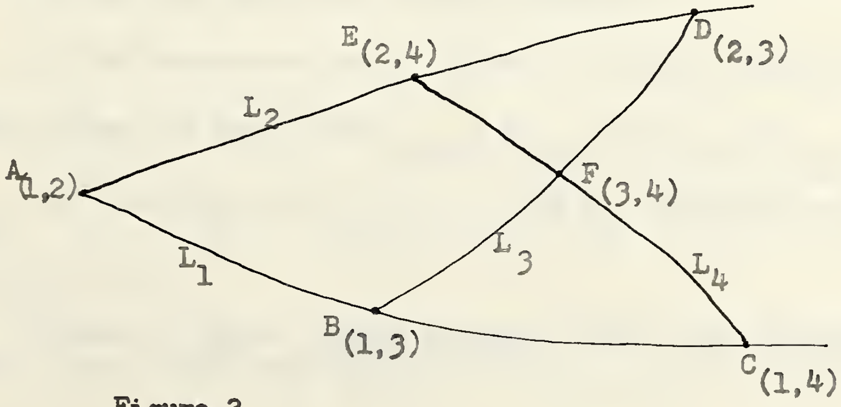

Figure 3

Note. The word "line" was left undefined and we are not concerned with ideas such as straight line or curved line. Whether a line as represented "appears" to be one or the other does not matter.

Definition. We shall define a new word, parallel, in this way. Points are parallel if they are not joined by a line.

Then a third theorem is possible.

T-3 Every point has exactly one point parallel to it.

The writer[8] of one modern geometry textbook gives another representation of this finite geometry using "bead" in place of "point," "wire" in place of "line," and "ornament" in place of "plane." These are undefined as are also the relations "bead in common" and "bead on a wire."

Exercise 1

(a) Restate the entire set of five postulates and two theorems (P'-1 to P'-5; T'-1 and T'-2) for the ornament. Define parallelism and state T'-3. Here is P'-1.

P'-1 Each pair of wires has at least one bead in common.

(b) How many beads are in this ornament?

(c) The representation of Figure 3 is also suitable for this ornament where the beads are $A_{(1,2)}$, $B_{(1,3)}$, $C_{(1,4)}$, $D_{(2,3)}$, $E_{(2,4)}$, and $F_{(3,4)}$, and the wires are L_1; L_2, L_3, and L_4. Name all of the possible

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[8] Claire F. Adler, Modern Geometry, (New York: McGraw-Hill Book Company, Inc., 1958), pp. 17-18.

sets of parallel beads.

(d) Why does the representation in Figure 4 not satisfy the postulates P-1 through P-5? (A, B, C, and D are points and a, b, c, and d are lines).

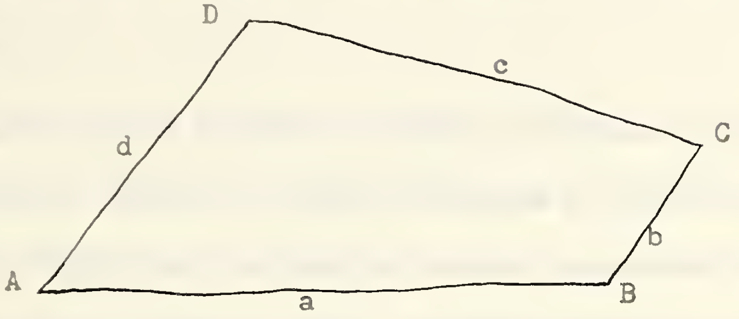

Figure 4

(e) Give a concrete representation which satisfies the following set of postulates. "Intersect" is another way of saying "lines on a point."[9]

1. There is at least one line on any two points.
2. There is at most one line on any two points.
3. There exists at least one line.
4. There are exactly two points on each line.
5. Corresponding to each line there is exactly one line that does not intersect it.

(f) Write a set of postulates for which a triangle can be a satisfactory representation.

(g) If from postulates P-1 through P-5 you exclude P-4 would Figure 3 still serve as a representation for the remaining set of four

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[9] Bruce E. Meserve, Fundamental Concepts of Geometry, (Cambridge, Mass: Addison-Wesley Publishing Co., Inc., 1955), p. 24.

postulates? Can you show another representation that does not correspond to Figure 3 but does satisfy the four postulates mentioned here? Is this set of four postulates consistent?

Duality

If, in the finite geometry of points and lines, we put the word "line" in place of "point" and "point" in place of "line" we obtain a new geometry. This is not the same as another representation of the same finite geometry (Exercise 1a). The table summarizes the terms and their replacements.

Finite Geometry I	Finite Geometry II
plane	plane
point	line
line	point

Exercise 2

(a) Using the terms for the new finite geometry as just explained, rewrite the postulates and theorems of the geometry first introduced in this unit.

(b) Show a representation of this geometry.

(c) The first finite geometry contained four lines and six points. How many lines and how many points has this geometry?

Note. This second finite geometry in a plane is a dual of the first finite geometry in a plane. Every time there is a point in I there is a line in II and every time there is a line in I there is a point in II. In I two lines have one and only one point on them. In

II two points have one and only one line on them.

Non-Euclidean Geometry

We have seen that beginning with different sets of postulates one can obtain different geometries. By considering a few finite geometries we have been able to see this more readily. Geometry as you study it in your regular class is developed from certain basic postulates. One of these was stated before in this unit.

> On a given point P, not on a line m, there exists one and only one line parallel to m.

You know that this geometry is essentially the same as that developed by Euclid and we call it Euclidean geometry. Concerning the one postulate just stated there is a story extending over centuries in the history of the development of mathematics. It relates the attempts of different mathematicians to prove this postulate. They had the idea it ought to be possible to establish it as a theorem. This they never did and in the process certain postulates were formulated to replace this one. This led to the discovery of some new geometries. Read this story in at least one of the following accounts or in another history of mathematics and proceed to exercise 3.

An Introduction to the History of Mathematics, Howard Eves and Carrol V. Newsom, pp. 62-72.

Prelude to Mathematics, W. W. Sawyer, Chapter 6.

Monographs on Topics of Modern Mathematics, J. W. A. Young, Chapter III.

Exercise 3

Complete all of the table which is designed to point out some major differences between Euclidean and the two non-Euclidean geometries.

A Comparison of Three Geometries

	Euclidean	Elliptical	Hyperbolic
Associated with what person?			
Through a point P not on line m, the number of lines that exist parallel to the line m is (are):			
The sum of the interior angles of any triangle is:			

Note. The two geometry systems that have just been compared with Euclidean geometry are named non-Euclidean. There exist also other systems of geometry which are described as not Euclidean.

References

Adler, Claire Fisher, Modern Geometry, McGraw-Hill Company, Inc., 1958, pp. 17-21.

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V LOGIC

Introduction

A statement is an assertion. Each of the following is a statement.

Pandit Nehru is the Prime Minister of India.

It is not raining today.

There will be a soccer game tomorrow.

Bombay is the capital of India.

$3 + 4 = 10$.

A statement may be true or false and it is this truth value (true or false) of a statement that is our concern in a brief study of logic. In this unit you will be attempting to determine the truth value of compound statements from the truth values of the simple statements that make up the compound statements.

Symbols

In order to be systematic and concise we use symbols, p, q, r, etc., to represent simple statements such as those above. Compound statements are formed when simple statements are joined by a connective. There are five connectives. These are "and," "or," "if..., then," "if and only if," and "not". The symbols for these are $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\sim$. Let p be the statement "Irene is a blonde". Let q be the statement "Ivan is a farmer". Here are five possible statements (four compound and one simple) in complete and symbolic form along with the

name that applies to each statement.

Irene is a blonde and Ivan is a farmer.	$p \wedge q$	conjunction
Irene is a blonde or Ivan is a farmer.	$p \vee q$	disjunction
If Irene is a blonde then Ivan is a farmer.	$p \rightarrow q$	implication
Irene is a blonde if and only if Ivan is a farmer.	$p \leftrightarrow q$	equivalence
Irene is not a blonde.	$\sim p$	negation

Note. "Not" ($\sim$) modifies a single statement but it is considered as a connective.

In mathematics these connectives have meanings more exact than their meanings in everyday speech. The disjunction has two meanings. These are the inclusive (p or q or both) and the exclusive (p or q but not both). If you say, "This afternoon I will go swimming or I will go to the library", it can mean you will do one or the other or that you will do both. This is the inclusive disjunction. If you say, "Gopal is a Bihari or a Bengali", you mean that he is one or the other but not both. This is the exclusive disjunction. In everyday speech you rely on intuition or the context to tell you which meaning is intended. In mathematics disjunction means the inclusive disjunction unless otherwise stated.

The negation of "Irene is a blonde" is not "Irene is a brunette," for she might be a red head or she might be platinum. The correct negation is "Irene is not a blonde," or "It is not the case that Irene is a blonde".

Exercise 1

Let p be the statement "Eunice is nice". Let q be the statement "Eunice is clever".

(a) Write in full the sentences indicated here in symbolic form.

(i) $\sim q$

(ii) $\sim p \wedge q$

(iii) $\sim p \vee \sim q$

(iv) $\sim\sim p$

(v) $\sim p \wedge \sim q$

(vi) $(\sim p \wedge q) \vee (p \wedge \sim q)$

(vii) $\sim p \rightarrow q$

(viii) $p \leftrightarrow \sim q$

(ix) $q \rightarrow p$

Truth Tables

The truth value of an individual statement may be represented by means of a truth table using F as the symbol for false and T as the symbol for true. If p is any statement its truth table is indicated in this way.

p
T
F

Let q be a second statement. Its truth table is as follows.

q
T
F

Since a single statement may be true or false two simple statements may be combined in four possible ways. We are interested in

the truth value of compound sentences and for each of the connectives these are defined by truth tables. The tables for conjunction and disjunction are given first.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

In the first table, when both simple statements are true the compound statement is also true. If p or q is false or if both p and q are false, the compound statement $p \wedge q$ is false.

Since the disjunction ($\vee$) is inclusive in mathematics the compound statement must be true in the first three cases of the second table. At least one of the simple statements must be true in order that the compound statement be true. Therefore the last case must be false in this table.

Here is the truth table for the statement of implication.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Consider p a hypothesis and q the conclusion. It seems reasonable that a true hypothesis can lead to a true conclusion and that

it cannot lead to a false conclusion. Thus the implication $(p \rightarrow q)$ is true in the first case and false in the second. What does a false hypothesis lead to? Actually it can lead to a true conclusion or a false one so we say that the reasoning in the implication is true in the third and fourth cases.

Let p be the statement "The shoemaker is repairing my shoes," and let q be the statement "This is an election year". The following sentences represent cases one, three, and four in the table and any one of them is true.

If the shoemaker is repairing my shoes then this is an election year.

If the shoemaker is not repairing my shoes then this is an election year.

If the shoemaker is not repairing my shoes then this is not an election year.

The truth table for equivalence follows.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Consider that $p \leftrightarrow q$ means the same as $p \rightarrow q$ and $q \rightarrow p$ which can be true only in the first and fourth cases. The reason is that T can never imply F and this is involved in cases two and three.

The final table, that for negation, has two entries only.

p	$\sim p$
T	F
F	T

Longer Compound Statements and Truth Tables

The five truth tables in the last section can be used to determine the truth value of compound statements that are more complicated. Study the examples.

p	q	p	$\wedge$	$\sim$	q
T	T	T	F	F	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	F	F	T	F
		1	3	2	1

p	q	p	$\rightarrow$	$(p$	$\vee$	$q)$
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	T	F	T	T
F	F	F	T	F	F	F
		1	3	1	2	1

Use the numbers below the tables to guide you. Wherever they appear give p and q the same truth value as in the first two columns (1). Then write the truth value under the less important connectives (2). (If there are parentheses begin from the inside and work out). Lastly determine the truth value for the principle connective (3) using the values for the most recent entries on either side of this connective. This column of final entry indicates the truth value of the compound statement.

Exercise 2

(a) Write truth tables for these compound statements.

(i) $\sim p \vee \sim q$

(ii) $(p \vee \sim q) \wedge \sim p$

(iii) $\sim(p \vee q) \wedge p$

(iv) $(p \wedge q) \rightarrow p$

(v) $(p \wedge q) \rightarrow [q \rightarrow (\sim p)]$

(vi) $(p \vee q) \leftrightarrow (q \vee p)$

(b) Let p be the statement "It is pouring rain". Let q be the statement "The river is going to flood". Write in symbols the following statements and determine their truth values by setting up truth tables.

If it is pouring rain then the river is going to flood.

If the river is going to flood then it is pouring rain.

The river is going to flood if and only if it is pouring rain.

If it is pouring rain then the river is not going to flood.

It is not the case that it is pouring rain if and only if the river is going to flood.

Logically True Statements

When a compound statement is true in every case we say that it is logically true. In the truth table only T appears under the principle connective.

Exercise 3

(a) Determine, using truth tables, how many of the following are logically true.

$\sim(p \wedge \sim p)$

$(p \rightarrow q) \wedge (q \rightarrow p) \leftrightarrow (p \leftrightarrow q)$

$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

$$[(p \rightarrow q) \wedge p] \rightarrow q$$

$$\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$$

(b) Which statements, if any, in Exercise 2a are logically true?

The Converse, Inverse and Contrapositive

You have learned that a theorem may have a converse, an inverse, and a contrapositive. You have learned also that the contrapositive of a theorem is always true. This can be demonstrated using rules of logic.

Exercise 4

(a) Let p be the statement "The opposite angles of a quadrilateral are equal". Let q be the statement "The quadrilateral is a parallelogram". Write the following in complete words.

The implication	$p \rightarrow q$
The converse	$q \rightarrow p$
The inverse	$\sim p \rightarrow \sim q$
The contrapositive	$\sim q \rightarrow \sim p$

(b) Complete this truth table which contains the three compound statements: the converse, the inverse, and the contrapositive.

p	q	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$

(c) We can say that the converse is sometimes true and that the contrapositive is logically true. What can be said about the inverse?

Inference

A mathematical system has terms that are undefined and postulates that must be assumed. From these, new definitions are obtained, and using the postulates, theorems are proved by logical methods. This proving of theorems is what we should now be able to do.

The fourth statement of Exercise 3a, $[(p \rightarrow q) \wedge p] \rightarrow q$, is logically true. It may be written in another form as shown below.

$p \rightarrow q$	If p, then q; and
$\underline{p}$	$\underline{p}$
$\therefore q$	imply q

Let p be "Kay is a junior," and let q be "Kay is studying biology".

$p \rightarrow q$	If Kay is a junior, then Kay is studying biology.
$\underline{p}$	Given: Kay is a junior.
$\therefore q$	Therefore Kay is studying biology.

Let p be "KLMN is a parallelogram," and let q be "The opposite sides of KLMN are equal".

$p \rightarrow q$	If KLMN is a parallelogram then its opposite sides are equal.
$\underline{p}$	Given: KLMN is a parallelogram
$\therefore q$	Therefore the opposite sides of KLMN are equal.

Now consider a statement that is not logically true. $[(p \rightarrow q) \wedge q] \rightarrow p$. Use a truth table to check this. Two examples follow.

Let p be "ABCD is a square," and let q be "ABCD is a rectangle".

$p \rightarrow q$ If ABCD is a square, then ABCD is a rectangle.

$\underline{q}$ Given: ABCD is a rectangle.

$\therefore p$ Therefore ABCD is a square.

Let p be "Triangle ABC is congruent to triangle DEF," and let q be "Triangle ABC is similar to triangle DEF".

$p \rightarrow q$ If triangle ABC is congruent to triangle DEF then triangle ABC is similar to triangle DEF.

$\underline{q}$ Given: Triangle ABC is similar to triangle DEF.

$\therefore p$ Therefore triangle ABC is congruent to triangle DEF.

In both of these examples the conclusion is false. By logic you have proved that the converse of an implication is not always true.

Exercise 5

(a) Let p be the statement "I do my homework," and let q be the statement "I shall pass this course". Determine whether the reasoning in the following is valid or not. Use truth tables if necessary.

(i) $p \rightarrow q$

$\underline{\sim p}$

$\therefore \sim q$

(ii) $p \rightarrow q$

$\underline{\sim q}$

$\therefore \sim p$

The Syllogism

To prepare a truth table for a compound statement such as

$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ it is necessary to use eight rows since there are three simple statements involved,--p, q, and r.

p	q	r	$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	
			1 2 1 3 1 2 1 4 1 3 1

Exercise 6

(a) Complete the columns in the order indicated using the two columns (3) to determine the final column (4). Note that the statement is always true. This is known in logic as the Law of the Syllogism.

(b) Let $(p \rightarrow q)$ be the statement "If two triangles are congruent, their corresponding angles are equal". Let $(q \rightarrow r)$ be the statement "If two triangles have their corresponding angles equal they are similar. Write in words the statement $(p \rightarrow r)$. Is this statement logically true?

(c) You have seen that the contrapositive of a statement is logically true. A statement of implication and its contrapositive are equivalent statements; they have the same truth value. There is a rule which says that a statement may be replaced by an equivalent statement.

Compare the statement that follows with the Law of the Syllogism.

$(\sim q \longrightarrow \sim p) \wedge (q \longrightarrow r) \longrightarrow (p \longrightarrow r)$

What has replaced $(p \longrightarrow q)$? Would you expect the compound statement to be valid? Test your answer using a truth table.

(d) Use the Law of the Syllogism as a basic pattern to set up the two following problems. First write in complete words the simple statements p, q, and r. Decide whether or not the reasoning is valid in each problem.

Given:

If the gardener is paid more, his son can go to school.

If the gardener's son cannot go to school, he cannot become a mail carrier.

Conclusion:

If the gardener is paid more, his son can become a mail carrier.

Given:

If it is raining, I will take my umbrella.

If I carry a lunch with me I will not take my umbrella.

Conclusion:

If it is raining I will not carry my lunch with me.

In this unit you have learned about the use of logic to test the validity of compound statements and you have seen how simple theorems may be proved logically with the use of symbols. This has been an introduction. You may sometime continue this study, learn more of the

rules, and see further how a mathematician goes about deciding whether or not a statement is "true".

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VI FINITE ARITHMETIC AND CONGRUENCES

A Finite Arithmetic

In arithmetic modulo 3 there is a finite number system of only three elements--the numbers 0, 1, and 2. Every problem in addition, subtraction, multiplication, or division with these numbers in this finite system will give a result that is one of these three numbers. Under these operations the system is closed. There are no fractions and no negative numbers in the system. The numbers have no order since no one number is greater or smaller than another. It is possible to solve algebraic equations in the system.

Exercise 1

(a) Complete the addition and multiplication tables modulo 3 for use in parts (b) and (c) of this exercise.

+	0	1	2
0			
1			
2			

x	0	1	2
0			
1			
2			

(b) Subtraction can be regarded as the inverse of addition. 2 - 1 = 1 since 1 + 1 = 2. Complete these problems in subtraction.

0 - 1 = (mod 3)   0 - 2 = (mod 3)   1 - 2 = (mod 3)

(c) Division is regarded as the inverse of multiplication. 2 ÷ 2 = 1 since 1 x 2 = 2. Complete these problems in division.

$2 \div 1 = \qquad$ (mod 3)

$1 \div 2 = \qquad$ (mod 3)

$0 \div 2 = \qquad$ (mod 3)

$1 \div 1 = \qquad$ (mod 3)

The Properties of a Field

There are four characteristics that describe a finite system such as the arithmetic mod 3.

1. If any number a is added to 0, the result is that number.

   $1 + 0 = 1$. "0" is said to be the identity element for addition.

   If any number a is multiplied by 1, the result is that number.

   $1 \times 2 = 2_{(\text{mod } 3)}$. "1" is the identity element for multiplication.

2. Each number other than 0 has a unique negative, a number which when added to it will give 0. $1 + 2 = 0_{(\text{mod } 3)}$. 1 is the negative of 2.

3. Each number other than 0 has a unique reciprocal. Two numbers whose product is 1 are reciprocals of one another. Since $2 \times 2 = 1_{(\text{mod } 3)}$, 2 is its own reciprocal.

4. Addition and multiplication are associative and commutative in the system. Multiplication is distributive with respect to addition.

In mathematics a system with these properties is called a field.

Exercise 2

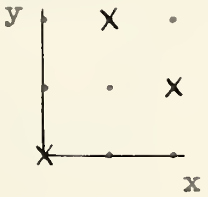
(a) Let us solve the equation $2x + 1 = 2_{(\text{mod } 3)}$.

First add the negative of 1. $\qquad 2x = 1_{(\text{mod } 3)}$

Then multiply by the reciprocal of 2. $x = 2_{(\text{mod } 3)}$

Similarly solve these equations: $x + 2 = 2$, $x + 1 = 2$.

(b) To graph a "line" we use a 9-point lattice and find that a graph of a linear equation consists of 3 points. The graph of $x + y = 0_{(\text{mod } 3)}$ consists of the set of points (0,0), (1,2), and (2,1).



$x + y = 0_{(\text{mod } 3)}$

Graph these "lines" each on its own lattice.

$2x + y = 1_{(\text{mod } 3)}$ $y = 1_{(\text{mod } 3)}$

$x + y = 1_{(\text{mod } 3)}$ $x = 2_{(\text{mod } 3)}$

(c) There ought to be a finite number of linear equations of the type $ax + by = c_{(\text{mod } 3)}$. Can you find all of them? Notice that $2x + y = 0_{(\text{mod } 3)}$ is the same as $x + 2y = 0_{(\text{mod } 3)}$ since the first equation multiplied by 2 will yield the second one.

Exercise 3

Use tables for addition and multiplication in the finite arithmetic modulo 5 in attempting these problems.

(a) $1 - 3 =$ (mod 5) $3 - 4 =$ (mod 5)

$0 - 2 =$ (mod 5)

(b) $3 \div 2 =$ (mod 5) $2 \div 3 =$ (mod 5)

$1 \div 4 =$ (mod 5) $0 \div 4 =$ (mod 5)

(c) Find the value of x.

$2x + 1 = 3_{(\text{mod } 5)}$, $3x + 2 = 3_{(\text{mod } 5)}$, $3x + 4 = 0_{(\text{mod } 5)}$

Residue Classes and Congruences

If the integers are divided by three and the remainders only are kept, the result is that all of the integers are divided into three classes: those with remainder 0, those with remainder 1, and those with remainder 2.

$0 = \{\ldots -3, 0, 3, 6, \ldots\}$

$1 = \{\ldots -2, 1, 4, 7, \ldots\}$

$2 = \{\ldots -1, 2, 5, 8, \ldots\}$

These are called residue classes. Any two numbers in the same class are congruent. Thus $4 \equiv 10_{(\text{mod } 3)}$.

Stating this in a different way, 3 will divide $(10 - 4)$ evenly, or, $(10 - 4)$ is divisible by 3.

Generally the congruence $a \equiv 0_{(\text{mod } m)}$ tells us that m divides a evenly.

In some ways congruences are like equations. Here are a few theorems that deal with congruences. They are stated without proof.

1. If $b \equiv a_{(\text{mod } m)}$, and $c \equiv a_{(\text{mod } m)}$;

   then $b \equiv c_{(\text{mod } m)}$.

2. If $a \quad b_{(\text{mod } m)}$, and $c \equiv d_{(\text{mod } m)}$,

   then $a + c \equiv b + d_{(\text{mod } m)}$,

   and $a - c \equiv b - d_{(\text{mod } m)}$,

   and $ac \equiv bd_{(\text{mod } m)}$.

Exercise 4

(a) In modulo 3 residue classes what is the sum of any member of residue class 2 and of residue class 1? What is the product?

(b) The congruence $86c \equiv 7_{(\text{mod } 11)}$ can be replaced by $9c \equiv 7_{(\text{mod } 11)}$ since $77c \equiv 0_{(\text{mod } 11)}$. Find congruences for the following that have coefficients smaller than the modulus.

$83x \equiv 7_{(\text{mod } 13)}$, $49x \equiv 283_{(\text{mod } 20)}$.[10]

(c) Find the least positive integers modulo 7 to which the integers 22, 312, and (22)(312) are congruent. This problem requires that you find the remainder when 22, 312, and (22)(312) are divided by 7, or, that you find x where $x \equiv 22_{(\text{mod } 7)}$ and $0 \leq x \leq 7$, and similarly for 312 and (22)(312).[11]

(d) For the last part of Theorem 2 on the previous page here is a corollary.

If $a \equiv b_{(\text{mod } m)}$, then $a^2 \equiv b^2_{(\text{mod } m)}$, and $a^k \equiv b^k_{(\text{mod } m)}$.
Find the remainder when 3^{10} is divided by 5.
$3^{10} \equiv (3^3)^3(3) \equiv (27)^3(3) \equiv (2^3)(3) \equiv (8)(3) \equiv 4_{(\text{mod } 5)}$. Therefore $3^{10} \equiv 4_{(\text{mod } 5)}$. Find the remainder when each of these numbers is divided by 51: 7^{10}, 3^{10}.

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[10] J. W. A. Young, Monographs on Topics of Modern Mathematics, (New York: Dover Publications, Inc., 1955), p. 322.

[11] Marie J. Weiss & Roy Dubisch, Higher Algebra for the Undergraduate, Second Edition, (New York: John Wiley & Sons, Inc., 1962), p. 19.

(e) Prove that $2^{11} - 1$ has the factor 23.

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VII POSSIBLE ADDITIONAL STUDIES

For the student who is able to continue special studies, a challenging topic would be topology using Topology The Rubber Sheet Geometry in the series Exploring Mathematics on Your Own, by Donovan Johnson and William Glenn, published by the Webster Publishing Company, 1960. Along with this Chapters I and II of Graphs and Their Uses by Oystein Ore in the New Mathematics Library of the School Mathematics Study Group may be read. The student may also do work on a programmed unit such as Groups and Fields, A Programmed Unit in Modern Mathematics, Early Boyd, William Moore and Wendell Smith, McGraw-Hill Book Co., 1963. Additional reading in several others of the volumes of the New Mathematics Library would probably appeal to the gifted student who has the time and whose interest in such reading remains high.

CHAPTER V

SUMMARY

A plan for introducing and implementing special provision for gifted students in mathematics in grades nine and ten in a small high school in the form of enrichment studies has been developed in this thesis. In particular the plan is set up for Woodstock School which is a school for English speaking foreign children in India.

Special provision for gifted children who are intellectually highly endowed and who are found generally in the upper ten per cent of the population according to I. Q. ratings is recommended because they are capable of doing more difficult or more advanced work and they are not sufficiently challenged by the regular studies given to the average students. It is stated that equal educational opportunity for all does not mean identical treatment for all but equal opportunity for all to develop their abilities to their greatest potential.

To identify gifted children requires more information than an I. Q. rating. An important item is the teacher's recommendation on the basis of the characteristics of the student. These deal with study habits, initiative, perseverance; and various mental traits such as ability to generalize, see relationships and think logically; insight; curiosity; a good memory; dislike for routine procedures and drill. In addition previous grades may be taken into account along with a score on a standardized mathematics achievement test.

Provision by means of enrichment studies was chosen because ability grouping is not possible in a small high school where there is usually just one class in a particular subject at each grade level. Acceleration is a highly recommended practice but it is also opposed because it does not appear to be a case of equal opportunity for all when the gifted study the same course as that followed by everyone and miss many extra topics that might be profitable to them. Further, by being accelerated, they receive less than their share of the total time that the school ought to give to them.

There is some dispute as to what enrichment is, for it seems to be applied sometimes to out-of-class activity where nothing of value is accomplished. It must include learning that is in depth and breadth beyond or in advance of the regular classroom study. Merely assigning more difficult problems of the same kind as those done in class does not pass for enrichment in mathematics. Topics included in descriptions of enrichment studies at the early high school level usually include portions of modern mathematics associated with the regular classroom study such as finite geometry, non-Euclidean geometry, number theory, mathematical systems, and selections from the history of mathematics.

Experimental support for enrichment in the literature is scarce but there seems to be some evidence to indicate that when students have had opportunity for enrichment they perform better on achievement tests and show a more positive attitude toward and interest in mathematics.

For Woodstock School in India criteria have been outlined in

this thesis for the staff, for the school, for the studies, and for the students.

The mathematics teacher must have adequate but not necessarily superior training.

The school must provide sufficient material in the way of extra textbooks and references in the library along with some of the enrichment reading materials that have become available in the past five years.

The studies must be outlined so that the student may proceed on his own without the teacher's help. Often the work must be done outside of class as the teacher's time in class is taken up with regular work and with assisting slower students. The studies should add to the student's total grasp and understanding of the field of mathematics.

The students must meet certain requirements in an I. Q. rating or aptitude rating, in a standardized mathematics achievement rating, in previous grades, and they must have the recommendations of the teachers in order to be permitted to participate in the studies.

The following is the list of six topics that have been developed in this thesis for the purpose of enabling the student to begin a program of enrichment studies in mathematics in grade nine or ten in Woodstock School.

The Pythagorean Theorem and Pythagorean Triples

A Consideration of π

India's Contribution to Mathematics
Finite Geometry and Non-Euclidean Geometry
Logic
Finite Arithmetic and Congruences

A student who is able to continue special studies would be directed to read other topics in recently available enrichment materials.

Probably as many as one-fourth of the students in some of the mathematics classes in grades nine and ten in Woodstock School may be permitted to participate in enrichment studies as suggested here. It should be a worthwhile experience to the student whether or not he has been privileged to have enrichment in mathematics previous to grade nine. After grade ten the brighter students may participate in the Advanced Placement Program in Mathematics especially if they plan to continue studying mathematics in college.

Enrichment studies in mathematics will probably continue to undergo changes in the future similar to what is now taking place. Several topics appearing in the literature ten years ago as enrichment are now appearing in the new mathematics programs recently prepared for use with all students in the classroom. Extra enrichment materials for the gifted student will probably tend to include more material of modern mathematics.

Because of the lack of materials prepared for individual use by brighter students in the heterogeneous classroom perhaps further efforts along this line could be pursued by others.

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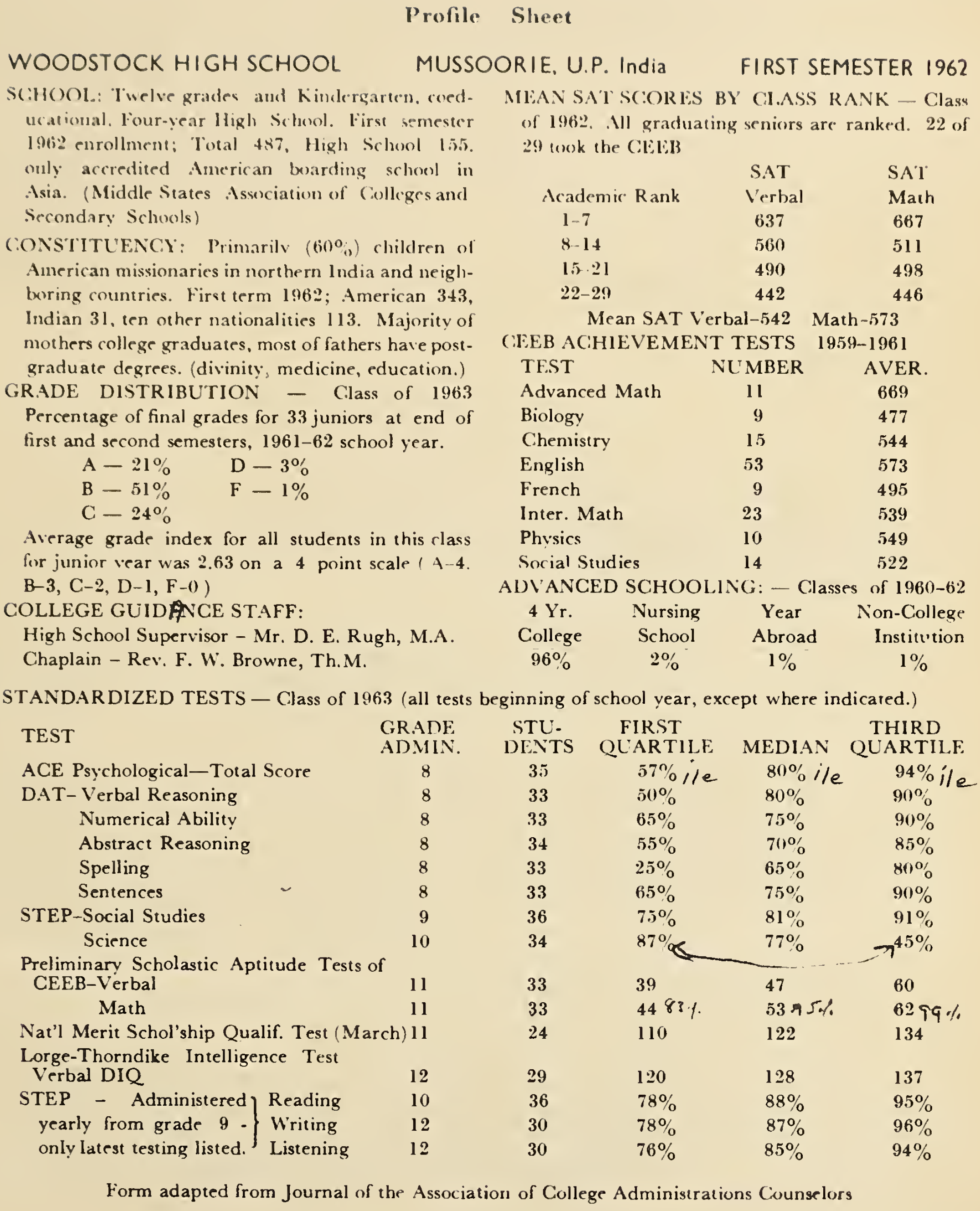
APPENDIX



Profile Sheet

WOODSTOCK HIGH SCHOOL MUSSOORIE, U.P. India FIRST SEMESTER 1962

SCHOOL: Twelve grades and Kindergarten, coeducational. Four-year High School. First semester 1962 enrollment; Total 487, High School 155. only accredited American boarding school in Asia. (Middle States Association of Colleges and Secondary Schools)

CONSTITUENCY: Primarily (60%) children of American missionaries in northern India and neighboring countries. First term 1962; American 343, Indian 31, ten other nationalities 113. Majority of mothers college graduates, most of fathers have post-graduate degrees. (divinity, medicine, education.)

GRADE DISTRIBUTION — Class of 1963 Percentage of final grades for 33 juniors at end of first and second semesters, 1961–62 school year.

A — 21% D — 3%
B — 51% F — 1%
C — 24%

Average grade index for all students in this class for junior year was 2.63 on a 4 point scale (A–4. B–3, C–2, D–1, F–0)

COLLEGE GUIDANCE STAFF:
High School Supervisor – Mr. D. E. Rugh, M.A.
Chaplain – Rev. F. W. Browne, Th.M.

MEAN SAT SCORES BY CLASS RANK — Class of 1962. All graduating seniors are ranked. 22 of 29 took the CEEB

Academic Rank	SAT Verbal	SAT Math
1–7	637	667
8–14	560	511
15–21	490	498
22–29	442	446

Mean SAT Verbal–542 Math–573

CEEB ACHIEVEMENT TESTS 1959–1961

TEST	NUMBER	AVER.
Advanced Math	11	669
Biology	9	477
Chemistry	15	544
English	53	573
French	9	495
Inter. Math	23	539
Physics	10	549
Social Studies	14	522

ADVANCED SCHOOLING: — Classes of 1960–62

4 Yr. College	Nursing School	Year Abroad	Non-College Institution
96%	2%	1%	1%

STANDARDIZED TESTS — Class of 1963 (all tests beginning of school year, except where indicated.)

TEST		GRADE ADMIN.	STUDENTS	FIRST QUARTILE	MEDIAN	THIRD QUARTILE
ACE Psychological—Total Score		8	35	57% i/e	80% i/e	94% i/e
DAT– Verbal Reasoning		8	33	50%	80%	90%
Numerical Ability		8	33	65%	75%	90%
Abstract Reasoning		8	34	55%	70%	85%
Spelling		8	33	25%	65%	80%
Sentences		8	33	65%	75%	90%
STEP–Social Studies		9	36	75%	81%	91%
Science		10	34	87%	77%	45%
Preliminary Scholastic Aptitude Tests of CEEB–Verbal		11	33	39	47	60
Math		11	33	44 83%	53 95%	62 99%
Nat'l Merit Schol'ship Qualif. Test (March)		11	24	110	122	134
Lorge-Thorndike Intelligence Test Verbal DIQ		12	29	120	128	137
STEP – Administered yearly from grade 9 - only latest testing listed.	Reading	10	36	78%	88%	95%
	Writing	12	30	78%	87%	96%
	Listening	12	30	76%	85%	94%

Form adapted from Journal of the Association of College Administrations Counselors

FIGURE 5

Woodstock High School Profile

WOODSTOCK TESTING PROGRAM

1962 - 1963

Grade	General Ability	Achievement	Inventories	Others
7	Lorge-Thorndike Intelligence 4V J	Iowa Tests of Basic Skills I J		
8	*ACE Psychological Examination	Iowa Tests of Basic Skills II J		
9	Differential Aptitude Tests J	STEP Reading 2A J STEP Writing 2A J STEP Listening 2A J	SRA Inventory J	
10		STEP Reading 2B J STEP Writing 2B J STEP Listening 2B J		STEP Science 2A J
11		STEP Reading 1A J STEP Writing 1A J STEP Listening 1A J	Kuder Preference Inventory J	PSAT Oct. NMSQT Mar. STEP Social Studies 2A J
12	Lorge-Thorndike Intelligence 5V	STEP Reading 1B J STEP Writing 1B J STEP Listening 1B J	SRA Youth Inventory	CEEB SAT Aug. CEEB Ach. Dec.

Woodstock is a member of the Educational Records Bureau. In addition to the above, Cooperative Tests from Educational Testing Service are administered in the high school (9-12) at the end of the school year in all mathematics and language classes. Tests are also administered which are published by the Physical Science Study Committee (PSSC) and the Chemical Education Material Study (CHEM).

*Tests thus marked are given to new students transferring into the high school.

KEY STEP - Sequential Tests of Educational Progress
NMSQT - National Merit Scholarship Qualifying Tests
CEEB - College Entrance Examination Board
PSAT - Preliminary Scholastic Aptitude Tests
J - July

Revised by departmental supervisors, July 1962.

FIGURE 6

Testing Program for Upper Grades of Woodstock School

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